Final Exam

1. T or F: (30%)

(a) The rank of an *m*x*n* matrix is at most max(*m*, *n*).

(b) The function $\det : M_{nxn}(F) \rightarrow F$ is a linear transformation only when *n*=2.

(c) Every linear operator always has the eigenvector $v = 0$.

(d) Every linear operator always has infinite eigenvectors.

(e) A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if V is the direct sum of the eigenspaces of T.

(f) Vectors *x* and *y* are independent if they are orthogonal.

(g) Every linear operator on a finite-dimensional inner product space has an adjoint.

(h) Let T be a unitary operator on a finite-dimensional inner product space V. If T is unitary, then there exists an orthogonal basis for V consisting of eigenvectors of T.

(i) Every self-adjoint operator is normal.

(j) Every orthogonal operator T in a finite-dimensional inner product space V satisfies

$\|T(x)\| = x$ for all $x \in V$.

Ans.:

(a) F

(b) F

(c) F

(d) F

(e) T

(f) F

(g) T

(h) T

(i) T

(j) T

2. Prove that the eigenvalues of a lower triangular matrix A are the diagonal entries of A. (8%)

Ans.:

Theorem 4.11. If *A* is a triangular *n*x*n* matrix, then det(*A*) $= A_{11}A_{22}\ldots A_{nn}$; that is, the determinant of *A* is the product of the entries of *A* that lie on the diagonal.

PROOF. Let *A* be a lower triangular *n*x*n* matrix. The proof is by induction on *n*. If *n*=2, then *A* has the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

And so det(A) $=A_{11}A_{22}-A_{12}\cdot 0=A_{11}A_{22}$, proving the theorem for lower triangular matrices if n=2.

Assume that the theorem is true for lower triangular nxn matrix. Then A has the form

$$\begin{pmatrix} A_{11} & 0 & \cdots & 0 & 0 \\ A_{21} & A_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{n(n-1)} & A_{nn} \end{pmatrix}$$

Expanding along the first row, we see that

$$\det(A)=A_{11}\cdot \det\begin{pmatrix} A_{22} & \cdots & 0 & 0 \\ A_{32} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A_{n2} & \cdots & A_{n(n-1)} & A_{nn} \end{pmatrix}=A_{11}\cdot (A_{22}\ldots A_{nn})$$

by the induction hypothesis. This completes the induction and proves the theorem for lower triangular matrices.

If A is a upper triangular matrix, then A^t is an lower triangular matrix. Hence the first part of this proof and Theorem 4.10 imply that
$\det(A)=\det(A^t)=(A^t)_{11}\ldots(A^t)_{nn}.=A_{11}\ldots A_{nn}$.

3. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. Prove that ant T-invariant subspaces of V containing v also contains W. (10%)

Ans.:

4. Show that $\forall x, y \in \mathrm{V},\ \ \|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$. (8%)

Ans.:

$$\because \|x+y\|^2=\langle x+y,x+y\rangle=\langle x,x+y\rangle+\langle y,x+y\rangle=\langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle$$

$$\|x-y\|^2=\langle x-y,x-y\rangle=\langle x,x-y\rangle-\langle y,x-y\rangle=\langle x,x\rangle-\langle x,y\rangle-\langle y,x\rangle+\langle y,y\rangle$$

$$\therefore \|x+y\|^2+\|x-y\|^2=2\langle x,x\rangle+2\langle y,y\rangle=2\|x\|^2+2\|y\|^2$$

5. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$ and $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$. (10%)

Ans.:

(a)

(i) $\forall x \in (W_1 + W_2)^{\perp}$

For any $w_1 \in W_1$, $w_2 \in W_2$

$x \perp (w_1 + w_2)$

$\therefore x \perp w_1 (w_2 = 0)$ and $x \perp w_2 (w_1 = 0)$

$\therefore x \perp W_1$ and $\therefore x \perp W_2$

$\therefore x \in W_1^{\perp} \cap W_2^{\perp}$

$\therefore (W_1 + W_2)^{\perp} \subseteq W_1^{\perp} \cap W_2^{\perp}$

(ii) $\forall x \in W_1^{\perp} \cap W_2^{\perp}$

For any $y \in W_1 + W_2$, $y = y_1 + y_2$, where $y_1 \in W_1$ and $y_2 \in W_2$

$\therefore \langle x, y \rangle = \langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle = 0 \Rightarrow x \perp y$

$\therefore x \in (W_1 + W_2)^{\perp}$

$\therefore W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1 + W_2)^{\perp}$

According to (i) and (ii), $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

Q.E.D.

(b)

$\because (W^{\perp})^{\perp} = W$

$\therefore W_1^{\perp} + W_2^{\perp} = \left(\left(W_1^{\perp} + W_2^{\perp}\right)^{\perp}\right)^{\perp} = \left(\left(W_1^{\perp}\right)^{\perp} \cap \left(W_2^{\perp}\right)^{\perp}\right)^{\perp}$ (by (a))

$= \left(W_1 \cap W_1\right)^{\perp}$

$\therefore (W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$

Q.E.D.

6. Solve $\begin{cases} 3x_1 + x_2 + x_3 = 4 \\ -2x_1 - x_2 = 12 \\ x_1 + 2x_2 + x_3 = -8 \end{cases}$ by using (a) Gaussian elimination and (b) Cramer's rule.

(10%)

Ans.:

(a) Gaussian elimination:

$$\begin{pmatrix} 3 & 1 & 1 & 4 \\ -2 & -1 & 0 & 12 \\ 1 & 2 & 1 & -8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & -8 \\ -2 & -1 & 0 & 12 \\ 3 & 1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & -8 \\ 0 & 3 & 2 & -4 \\ 0 & -5 & -2 & 28 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 2 & 1 & -8 \\ 0 & 3 & 2 & -4 \\ 0 & -2 & 0 & 24 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & -8 \\ 0 & 3 & 2 & -4 \\ 0 & 1 & 0 & -12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 16 \\ 0 & 0 & 2 & 32 \\ 0 & 1 & 0 & -12 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_3 = 16 \\ 2x_3 = 32 \\ x_2 = -12 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = -12 \\ x_3 = 16 \end{cases}$$

(b) Cramer's rule:

$$\det(A) = \det\begin{pmatrix} 3 & 1 & 1 \\ -2 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = -4$$

$$\det(M_1) = \det\begin{pmatrix} 4 & 1 & 1 \\ 12 & -1 & 0 \\ -8 & 2 & 1 \end{pmatrix} = 0$$

$$\det(M_2) = \det\begin{pmatrix} 3 & 4 & 1 \\ -2 & 12 & 0 \\ 1 & -8 & 1 \end{pmatrix} = 48$$

$$\det(M_3) = \det\begin{pmatrix} 3 & 1 & 4 \\ -2 & -1 & 12 \\ 1 & 2 & -8 \end{pmatrix} = -64$$

$$\Rightarrow \begin{cases} x_1 = \dfrac{0}{-4} = 0 \\ x_2 = \dfrac{48}{-4} = -12 \\ x_3 = \dfrac{-64}{-4} = 16 \end{cases}$$

7. Evaluate the determinant of the matrix $\begin{pmatrix} 2 & -1 & 2 & 1 \\ -3 & 4 & 2 & -1 \\ 1 & 0 & 3 & 0 \\ -2 & 6 & -4 & 1 \end{pmatrix}$. (6%)

Ans.:

$$\begin{pmatrix} 2 & -1 & 2 & 1 \\ -3 & 4 & 2 & -1 \\ 1 & 0 & 3 & 0 \\ -2 & 6 & -4 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & -4 & 1 \\ 0 & 4 & 11 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & 6 & 2 & 1 \end{pmatrix},$$

$$\det\begin{pmatrix} 0 & -1 & -4 & 1 \\ 0 & 4 & 11 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & 6 & 2 & 1 \end{pmatrix} = (-1)\cdot\det\begin{pmatrix} -1 & -4 & 1 \\ 4 & 11 & -1 \\ 6 & 2 & 1 \end{pmatrix} = -31$$

8. V=P_2(R) and T is defined by T(f(x))=f(0)+ f(2)(x+x^2). Test T for diagonalizability, and find a basis β such that $[T]_\beta$ is diagonal, if possible. (10%)

Ans.:

Let standard order basis γ in V=P_2(R) is $\gamma = \{1, x, x^2\}$

$$\begin{aligned} T(1) &= 1 + 1(x + x^2) = 1 + x + x^2 \\ T(x) &= 0 + 2(x + x^2) = 2x + 2x^2 \\ T(x^2) &= 0 + 4(x + x^2) = 4x + 4x^2 \end{aligned} \Rightarrow [T]_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix}$$

$$\det[T - \lambda I] = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 4 \\ 1 & 2 & 4-\lambda \end{pmatrix} = \lambda(1-\lambda)(\lambda - 6) \Rightarrow \lambda = 0,1,6 \text{ (eigenvalues)}$$

$\Rightarrow$ T is diagonalizable.

$$\lambda = 0, T - \lambda I = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix} \Rightarrow eigenvector\ \ v_1 = \left\{ t\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix},\ t \in R \right\}$$

$$\lambda = 1, T - \lambda I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 4 \\ 1 & 2 & 3 \end{pmatrix} \Rightarrow eigenvector\ \ v_2 = \left\{ t\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix},\ t \in R \right\}$$

$$\lambda = 6, T - \lambda I = \begin{pmatrix} -5 & 0 & 0 \\ 1 & -4 & 4 \\ 1 & 2 & -2 \end{pmatrix} \Rightarrow eigenvector\ \ v_3 = \left\{ t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ t \in R \right\}$$

$$\Rightarrow \beta = \{(-2x + x^2), (-5 + x + x^2), (x + x^2)\}$$

$$\Rightarrow [\mathrm{T}]_{\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

9. For the inner product space V=R^3 over F, and linear transformations $g(a_1, a_2, a_3)=a_1+2a_2-4a_3$, find a vector y such that $g(x)=<x, y>$ for all $x \in V$

Ans.:

$$y = (1,2,-4)$$