Midterm

1. A real-valued function f defined on the real line is called an odd function if $f(-t)=-f(t)$ for each real number t ◦ Is the set of odd functions defined on the real line with the operations of addition and scalar multiplication, defined for two functions f, g and $c \in F$ by $(f+g)(t) = f(t)+g(t)$ and $(cf)(t)=c[f(t)]$, a vector space? Justify your answer ◦ (10%)

Ans.:

Let V denote the set of odd functions.

(1) Closure of vector addition:

For all $f(t)$, $g(t)$ in V, define $l(t) = f(t)+g(t)$

$\because l(t) = f(t) + g(t) = -\left[f(-t) + g(-t)\right] = -l(-t)$

$\therefore l(t) \in \mathrm{V}$

(2) Closure of scalar multiplication:

For each element a in F and all $f(t)$ in V,

$a(f(t)) = -a(f(-t)) \in \mathrm{V}$

(3)

For all $f(t)$, $g(t)$ in V, $f(t)+g(t)=g(t)+f(t)$ (commutativity of addition)

$\Rightarrow$ (VS 1) holds.

(4)

For all $f(t)$, $g(t)$ and $h(t)$ in V, $(f(t)+g(t))+h(t) = f(t)+g(t)+h(t)=f(t)+(g(t)+h(t))$ (associativity of addition)

$\Rightarrow$ (VS 2) holds.

(5)

Define $w(t)$ by $w(t) = 0$ for all t.

$\because w(t) = -w(-t) = 0$

$\therefore w(t)$ is an odd function.

$\Rightarrow w(t) \in \mathrm{V}$

For all $f(t)$ in V, $f(t)+w(t) = f(t)+0=f(t)$.

$\Rightarrow$ (VS 3) holds.

(6)

Define $g(t)$ by $g(t)=-f(t)$

$\because g(t)=-f(t)=-(-f(-t))=-g(-t)$ is an odd function.

$\therefore g(t) \in \mathrm{V}$

$\Rightarrow f(t)+g(t)=f(t)-f(t)=0$

$\Rightarrow$ (VS 4) holds.

(7)

For any $f(t)$ in V, $1 \cdot f(t)=f(t)$

$\Rightarrow$(VS 5) holds.

(8)

For each pair of elements a, b in F and any $f(t)$ in V,

$(ab)f(t)=a(bf(t))=a(bf\,(t)\,)$

$\Rightarrow$(VS 6) holds.

(9)For each element a in F and each pair of $f(t)$, $g(t)$ in V,

$a(f(t)+g(t)) = af(t)+ag(t)$

$\Rightarrow$(VS 7) holds.

(10)

For each pair of elements a, b in F and any $f(t)$ in V,

$(a+b)f(t)=af(t)+bf(t)$

$\Rightarrow$(VS 8) holds.

According to (1)~(10), the set of odd functions is a vector space.

2. In $M_{mxn}(F)$ define $W=\{A\in M_{mxn}(F): \sum_{i=1}^{m} A_{ij}=0$ whenever $1 \leqq j \leqq n\}$

(a)Is the set W a subspace of $M_{mxn}(F)$? Justify your answer。(6%)

(b)Find a basis for the smallest vector space that contain W。(6%)

Ans.:

(a) Suppose that there are any two vectors B and C in W, where $\sum_{i=1}^{m} B_{ij} = \sum_{i=1}^{m} C_{ij} = 0$.

Suppose that a is a scalar and $D = O_{mxn}$.

$\because \sum_{i=1}^{m}(aB+C)_{ij} = \sum_{i=1}^{m}(aB_{ij}+C_{ij}) = \sum_{i=1}^{m}(aB_{ij}) + \sum_{i=1}^{m} C_{ij} = a\sum_{i=1}^{m} B_{ij} + \sum_{i=1}^{m} C_{ij} = 0$ and $\sum_{i=1}^{m} D_{ij} = 0$

$\therefore aB+C \in W$, $D \in W$

$\Rightarrow$W is a subspace of $M_{mxn}(F)$

(b)

The basis contains $(m-1)\times n$ vectors. A basis for W is $\{A^{ij}: 1 \leqq i \leqq m\text{-}1,\ 1 \leqq j \leqq n\}$,
where A^{ij} is the $m\times n$ matrix having 1 in the ith row and jth column, -1 in the mth row and jth column, and 0 elsewhere.

3. Determine which of the following sets are bases for $P_2(R)$。(8%)

(a)$\{1+2x+x^2,3+x^2,x+x^2\}$ (b)$\{1+2x-x^2,4-2x+x^2,1+18x-9x^2\}$

Ans.:

(a) $\because$ The only solution of $a(1+2x+x^2)+b(3+x)+c(x+x^2)=0$ is $a=b=c=0$

$\therefore\{1+2x+x^2, 3+x, x+x^2\}$ is linearly independent.

$\because\dim(P_2(R)) = 3 =$ the number of vectors in the set $\{1+2x+x^2, 3+x, x+x^2\}$

According to Corollary 2 of Theorem 1.10, $\{1\text{-}2x\text{-}2x^2, -2+3x\text{-}x^2, 1\text{-}x+6x^2\}$ is a basis for $P_2(R)$.

(b) $\because$ The solution of $a(1+2x-x^2)+b(4-2x+x^2)+c(1+18x-9x^2)=0$ is $\{a, b, c\}=\{-37,8,5\}$

$\therefore \{1+2x-x^2, 4-2x-x^2, 1+18x-9x^2\}$ is not linearly independent.

$\Rightarrow \{1+2x-x^2, 4-2x-x^2, 1+18x-9x^2\}$ is not a basis for $P_2(R)$.

4. Let V be a finite-dimensional vector space and T: V $\to$ V be linear.

(a) Show that rank(T) = dim(V) – nullity(T). (8%)

(b) Suppose that V = R(T) + N(T). Show that V=R(T)$\oplus$N(T) based on (a). (8%)

Ans.:

(a) Let the dimension of V be n.

Suppose that $\nu = \{v_1, v_2, ..., v_m\}$ is a basis for N(T),

$\Rightarrow$ nullity(T) = m.

According to Corollary of Theorem1.11 in Sec. 1.6, by extending ν, we have $\alpha = \{v_1, v_2, ..., v_m, v_{m+1}, ..., v_n\}$ which is a basis for V.

For any vector u in R(T),

$$u = \sum_{i=1}^{n} a_i \mathrm{T}(v_i) = \sum_{i=1}^{m} a_i \mathrm{T}(v_i) + \sum_{i=m+1}^{n} a_i \mathrm{T}(v_i) = \sum_{i=m+1}^{n} a_i \mathrm{T}(v_i)$$

$\Rightarrow \{\mathrm{T}(v_{m+1}), ..., \mathrm{T}(v_n)\}$ generates R(T)

Then we suppose that

$$\sum_{i=m+1}^{n} b_i \mathrm{T}(v_i) = 0 \quad \text{for } b_{m+1}, b_{m+2}, \ldots, b_n \in \mathrm{F}.$$

Using the fact that T is linear, we have

$$\mathrm{T}\left(\sum_{i=m+1}^{n} b_i v_i \right) = 0$$

$$\therefore \sum_{i=m+1}^{n} b_i v_i \in \mathrm{N(T)}$$

Hence there exist $c_1, c_2, \ldots, c_m \in$ F such that

$$\sum_{i=m+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i \text{ or } (-\sum_{i=1}^{k} c_i v_i) + \sum_{i=m+1}^{n} b_i v_i = 0$$

Since α is a basis for V, we have $b_i = 0$ for all i.

Hence $\{\mathrm{T}(v_{m+1}), ..., \mathrm{T}(v_n)\}$ is linear independent.

$\Rightarrow \{\mathrm{T}(v_{m+1}), ..., \mathrm{T}(v_n)\}$ is a basis for R(T).

$\Rightarrow$ rank(T) = n-m = dim(V) – nullity(T).

Q.E.D.

(b) Suppose that rank(T) = dim(V)-nullity(T) and V=R(T)+N(T), then according to dimension theorem:

dim(V) = dim(R(T)) + dim(N(T)) - dim(R(T)$\cap$N(T))……………………………...(A)

where dim(R(T)) = rank(T) and dim(N(T)) = nullity(T).

But rank(T) = dim(V)-nullity(T)

$\Rightarrow$dim(V)= rank(T)+nullity(T) $\Rightarrow$dim(V)= dim(R(T))+dim(N(T)…………………(B)

Compare (A) and (B), we know dim(R(T)$\cap$N(T)) = 0

$\Rightarrow$R(T)$\cap$N(T)={0}

According to V=R(T)+N(T) and R(T)$\cap$N(T)={0}, V=R(T)$\oplus$N(T)

Q.E.D.

5. Define T: P(R)$\rightarrow$P(R) by $T(f(x)) = \int_0^x f(x)dt$. Prove that T is linear and one-to-one, but not onto. (10%)

Ans.:

(i)Suppose that $f(x)$ and $g(x)$ in P(R), a is a scalar, then

T($af(x)$+$g(x)$) = $\int_0^x (af(x)+g(x))dt = \int_0^x af(x)dt + \int_0^x g(x)dt =$

$a\int_0^x f(x)dt + \int_0^x g(x)dt$ = aT($f(x)$)+T($g(x)$)

$\therefore$T is linear.

(ii) $\because$There is no solution for function $\mathrm{T}(f(x)) = 2$.

$\therefore$T is not onto.

(iii) For any $f(x) \neq 0,\ T(f(x)) \neq 0$.

$\because$N(T)={ 0 }

$\therefore$ T is one-to-one.

6. For ordered based $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$, $\beta' = \{x^2, x+1, 1\}$ for P_2(R), find the change of coordinate matrix that changes β'-coordinates into β-coordinates. (10%)

Ans.: $\because$ $$\begin{aligned} x^2 &= 0\cdot(2x^2 - x) + 0\cdot(3x^2+1) + 1\cdot x^2 \\ x+1 &= -1\cdot(2x^2 - x) + 1\cdot(3x^2+1) + (-1)\cdot x^2 . \\ 1 &= 0\cdot(2x^2 - x) + 1\cdot(3x^2+1) + (-3)\cdot x^2 \end{aligned}$$

$\therefore$Coordinate matrix is $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -3 \end{pmatrix}$.

7. T: $R^3 \rightarrow R^2$ defined by T(a_1, a_2, a_3) = (a_1+a_2, 3a_3). Find bases for both N(T) and R(T). (10%)

Ans.:

(i)Suppose that (a, b, c) in N(T), then T(a, b, c)= (a+b,3c)={0}

$$\Rightarrow \begin{cases} a+b=0 \\ 3c=0 \end{cases} \Rightarrow \begin{cases} a=-b \\ c=0 \end{cases}$$

$\therefore$The basis for N(T) is {(1, -1, 0)}

(ii)For any element (d, e) in R^2, we can find an element (a, b, c) = (d, -d, e/3) in R^3, such that

T(a, b, c) = (d, e).

$\therefore R^2 \subseteq$ R(T)

$\because$ R(T) $\subseteq R^2$

$\therefore R^2 =$ R(T)

The basis for R(T) is also a basis for R^2.

$\therefore$The basis for R(T) is {(1, 0), (0, 1)}

8. Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, ..., v_n\}$. Define v_0 = 0.

There exist a linear transformation T: V$\rightarrow$V such that $T(v_j) = v_j + v_{j-1}$ for

j=1,2,…,n. Compute $[T]_\beta$. (10%)

Ans.:

$$T(v_1) = 1\cdot v_1 + 0\cdot v_2 + 0\cdot v_3 + ... + 0\cdot v_n$$
$$T(v_2) = 1\cdot v_1 + 1\cdot v_2 + 0\cdot v_3 + ... + 0\cdot v_n$$
$$T(v_3) = 0\cdot v_1 + 1\cdot v_2 + 1\cdot v_3 + ... + 0\cdot v_n$$
$$\vdots$$
$$T(v_n) = 0\cdot v_1 + 0\cdot v_2 + ... + 1\cdot v_{n-1} + 1\cdot v_n$$

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

9. Let V and W be finite-dimensional vector spaces and T: V$\rightarrow$W be linear. Label the following statements as true or false. (6%)

(a) Any union of subspaces of V is a subspace of V.

(b) If dim(V) < dim(W), then T can not be onto.

(c) If dim(V) > dim(W), then T can not be one-to-one.

Ans.:

(a)False. Suppose that V' and V'' are subspaces of V with basis {(1,0,0)} and {(0,1,0), (0,0,1)}.

$\because (1,1,1) = (1,0,0)+(0,1,0)+(0,0,1) \notin (V^{'} \cup V^{''})$

$\therefore (V^{'} \cup V^{''})$ is not a vector space$\Rightarrow$Statement is false.

(b)True. If dim(V)<dim(W), $R(T) \neq W$. $\Rightarrow$Statement is true.

(c)True. If dim(V)>dim(W), $N(T) \neq \{0\}$. $\Rightarrow$Not one-to-one. $\Rightarrow$Statement is true.

10. Let A=$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 0 \end{pmatrix}$ and B=$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 0 & 4 & -1 \end{pmatrix}$. Find an elementary matrix E such that

B = EA. (8%)

Ans.:

$$\because \quad (-1) \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 0 & 4 & -1 \end{pmatrix}$$

$$\therefore E = (-1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$