Solution 1

Sec. 1.2

4.

(a) $\begin{bmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 5 \\ -5 & 3 & 2 \end{bmatrix}$

(c) $4\begin{bmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{bmatrix} =$

Ans.:

(a) $\begin{bmatrix} 6 & 3 & 2 \\ -4 & 3 & 9 \end{bmatrix}$

(c) $\begin{bmatrix} 8 & 20 & -12 \\ 4 & 0 & 28 \end{bmatrix}$

12.

A real-valued function f defined on the real line is called an even function if $f(-t) = f(t)$ for each real number *t*. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

Ans.:

Let V denote the set of even functions.

(1) Closure of vector addition:

For all *f*(*t*), *g*(*t*) in V, define *l*(*t*) = *f*(*t*)+*g*(*t*)

$\because l(t) = f(t) + g(t) = f(-t) + g(-t) = l(-t)$

$\therefore l(t) \in \mathrm{V}$

(2) Closure of scalar multiplication:

For each element *a* in *F* and all *f*(*t*) in V,

$a(f(t)) = a(f(-t)) \in \mathrm{V}$

(3)

For all *f*(*t*), *g*(*t*) in V, *f*(*t*)+*g*(*t*)=*g*(*t*)+*f*(*t*) (commutativity of addition)

$\Rightarrow$ (VS 1) holds.

(4)

For all *f*(*t*), *g*(*t*) and *h*(*t*) in V, (*f*(*t*)+*g*(*t*))+*h*(*t*) =*f*(*t*)+*g*(*t*)+*h*(*t*)=*f*(*t*)+(*g*(*t*)+*h*(*t*)) (associativity of addition)

$\Rightarrow$ (VS 2) holds.

(5)

Define $w(t)$ by $w(t) = 0$ for all t.

$\because w(t) = w(-t) = 0$

$\therefore$ $w(t)$ is an even function.

$\Rightarrow$ $w(t) \in$ V

For all $f(t)$ in V, $f(t)+ w(t) =f(t)+0=f(t)$.

$\Rightarrow$(VS 3) holds.

(6)

Define $g(t)$ by $g(t)=-f(t)$

$\because g(t)=-f(t)=-f(-t)=g(-t)$ is an even function.

$\therefore$ $g(t) \in$ V

$\Rightarrow f(t)+g(t)=f(t)-f(t)=0$

$\Rightarrow$(VS 4) holds.

(7)

For any $f(t)$ in V, $1 \cdot f(t)=f(t)$

$\Rightarrow$(VS 5) holds.

(8)

For each pair of elements a, b in F and any $f(t)$ in V,

$(ab)f(t)=a(bf(t))=a(bf\,(t)\,)$

$\Rightarrow$(VS 6) holds.

(9)For each element a in F and each pair of $f(t)$, $g(t)$ in V,

$a(f(t)+g(t)) = af(t)+ag(t)$

$\Rightarrow$(VS 7) holds.

(10)

For each pair of elements a, b in F and any $f(t)$ in V,

$(a+b)f(t)=af(t)+bf(t)$

$\Rightarrow$(VS 8) holds.

According to (1)~(10), the set of even functions is a vector space.

18.

Let V=$\{(a_1, a_2): a_1, a_2 \in R\}$. For (a_1, a_2), $(b_1, b_2) \in$ V and $c \in R$, define

$(a_1, a_2) + (b_1, b_2) = (a_1+2b_1, a_2+3b_2)$ and $c(a_1, a_2) = (ca_1, ca_2)$.

Is V a vector space over R with these operations? Justify your answer.

Ans.:

$\because (a_1, a_2) + (b_1, b_2) = (a_1+2b_1, a_2+3b_2)$, $(b_1, b_2) + (a_1, a_2) = (b_1+2a_1, b_2+3a_2)$

$\Rightarrow$ $(a_1+2b_1, a_2+3b_2) \neq (b_1+2a_1, b_2+3a_2)$, (VS 1) fails to hold.

$\therefore$V is not a vector space

Sec. 1.3

5.

Prove that $A+A^t$ is symmetric for any square matrix A.

Ans.:

Let the entry of A that lies in row i and column j be a_{ij}. Then the entry of A^t that lies in row i and column j is a_{ji}.

$\Rightarrow$ The entry of matrix $(A+A^t)$ that lies in row i and column j is $a_{ij}+a_{ji}$. Similarly, $a_{ji}+a_{ij}$ lies in row j and column i of matrix $(A+A^t)$.

$\because$ $a_{ij}+a_{ji} = a_{ji}+a_{ij}$

$\therefore$ The entry of matrix $(A+A^t)$ that lies in row i and column j is equal to the entry of matrix $(A+A^t)$ that lies in row j and column i.

$\Rightarrow A+A^t$ is symmetric.

10.

Prove that $W_1 = \{(a_1,a_2,\ldots,a_n) \in F^n: a_1+a_2+\ldots+a_n = 0\}$ is a subspace of F^n, but $W_2 = \{(a_1,a_2,\ldots,a_n) \in F^n: a_1+a_2+\ldots+a_n = 1\}$ is not.

Ans.:

(I)

(i) For any two vector $x=(x_1, x_2,\ldots, x_n)$ and $y=(y_1, y_2,\ldots, y_n)$ in W_1,

$x_1 + x_2+\ldots+x_n = 0$

$y_1 + y_2+\ldots+y_n = 0$

$x + y = (x_1 + y_1, x_2 + y_2,\ldots, x_n + y_n)$

$\Rightarrow (x_1 + y_{1)} +(x_2 + y_2)+\ldots+ (x_n + y_n)$

$= (x_1 + x_2 +\ldots+ x_n) +(y_1 + y_2 +\ldots+ y_n)$

$= 0$

$\therefore x + y \in W_1$

(ii) For any $c \in F$

$cx = c(x_1, x_2,\ldots,x_n) = (cx_1, cx_2,\ldots, cx_n)$

$\Rightarrow cx_1+cx_2+\ldots+cx_n = c(x_1+x_2+\ldots+x_n) = c \cdot 0 = 0$

$\therefore cx \in W_1$

(iii) For zero vector $z = (0,0,\ldots,0) \in F^n$

$\because 0+0+\ldots+0 = 0$

$\therefore z \in W_1$

Base on (i), (ii) and (iii), W_1 is a subspace of F^n.

(II)

(i) For any two vector $x=(x_1, x_2,\ldots, x_n)$ and $y=(y_1, y_2,\ldots, y_n)$ in W_2, then

$x_1 + x_2+\ldots+x_n = 1$

$y_1 + y_2+\ldots+y_n = 1$

$x + y = (x_1 + y_1, x_2 + y_2,\ldots, x_n + y_n)$

$\Rightarrow$ $(x_1 + y_1)+ (x_2 + y_2)++ (x_n + y_n)$

$= (x_1 + x_2 + \ldots + x_n) + (y_1 + y_2 + \ldots + y_n)$

$= 1+1= 2 \neq 1$

$\therefore x+y \notin W_2$

(ii) For any $c \in F$ is a constant

$cx = c(x_1, x_2,\ldots,x_n) = (cx_1, cx_2,\ldots, cx_n)$

$\Rightarrow cx_1+cx_2+\ldots+cx_n = c(x_1+x_2+\ldots+x_n) = c \cdot 1 = c$

$cx \notin W_2$

(iii) For zero vector $z = (0,0,\ldots,0) \in F^n$

$\because 0+0+\ldots+0 = 0 \neq 1$

$\therefore z \notin W_1$

According to ether (i) or (ii) or (iii), W_2 is not a subspace of F^n.

19.

Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Ans.:

($\Leftarrow$) Suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_2$ or W_1,

$\because W_2$ and W_1 are subspace of V.

$\therefore W_1 \cup W_2$ is a subspace of V.

($\Rightarrow$) Assume $W_1 \cup W_2$ is a subspace of V:

Suppose that neither $W_1 \subseteq W_2$ nor $W_2 \subseteq W_1$ is true.

We can find two vector x and y which satisfy $x \in W_1$, $x \notin W_2$ and $y \in W_2$, $y \notin W_1$ $(x \neq y \neq 0)$.

$\because x \in W_1$ and $y \in W_2$

$\therefore x$ and $y \in W_1 \cup W_2$

$\because W_1 \cup W_2$ is a subspace of V

$\therefore x+ y \in W_1 \cup W_2$

$\Rightarrow$ $x+y \in W_1$ or W_2

a)

Suppose $x+y \in W_1$:

$\because x \in W_1$

$\therefore -x \in W_1$

$\Rightarrow (x+y)+(-x) \in W_1 \Rightarrow y \in W_1 \Rightarrow$ conflicting

b)

Suppose $x+y \in W_2$

$\because y \in W_1$

$\therefore -y \in W_2$

$\Rightarrow (x+y)+(-y) \in W_2 \Rightarrow \ x \in W_2 \Rightarrow$ conflicting

$\Rightarrow$ The assumption of "neither $W_1 \subseteq W_2$ nor $W_2 \subseteq W_1$ is true " fails.

$\therefore W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Base on (i) and (ii), $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \supseteq W_1$.

Q.E.D.

Sec. 1.4

5.

In each part, determine whether the given vector is in the span of *S*.

(b) (-1, 2, 1), *S* = {(1,0,2), (-1,1,1)}

(g) $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $S=\left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

Ans.:

(b) (-1, 2, 1) isn't in the span of S because there are't any solution for

$a(1,0,2)+b(-1,1,1)=(-1, 2, 1)$

(g) $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ is in the span of S because $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

13.

Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$ then span(S_1) $\subseteq$ span(S_2). In particular, if $S_1 \subseteq S_2$ and span(S_1) = V, deduce that span(S_2) = V.

Ans.:

(A)

Suppose that $S_1 = \{x_1, x_2, ..., x_m\}$ is a subset of a vector space V

$\Rightarrow$ Span(S_1) and Span(S_2) are subspace of V

For all $x \in$ Span (S_1), $x = a_1x_1 + a_2x_2 + ... + a_mx_m$

$\because S_1 \subseteq S_2$

$\therefore \{x_1, x_2, ..., x_m\} \in \text{Span}(S_2)$

$\Rightarrow x = a_1x_1 + a_2x_2 + ... + a_mx_m \in S_2$

$\Rightarrow$ Span(S_1) $\subseteq$ Span(S_2)

(B)

According to (A), if $S_1 \subseteq S_2$ then Span(S_1) $\subseteq$ Span(S_2).

$\because$ Span(S_1) =V

$\Rightarrow$ V $\subseteq$ Span(S_2)

According to Theorem 1.5,

if S_2 is a subset of the vector space V then Span(S_2) $\subseteq$ V

$\therefore$ Span(S_2) =V