Solution 2

Sec. 1.5

1.5.5. Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent.

Ans.:

For $\{1, x, x^2, ..., x^n\}$, the only solution of $a_1x + a_2x^2 + \ldots + a_nx^n = 0$ is $a_1 = a_2 = \ldots = a_n = 0$

$\Rightarrow \{1, x, x^2, ..., x^n\}$ is linearly independent

1.5.9. Let u and v be distinct vector in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Ans.:

$(\Rightarrow)$

Suppose $\{u, v\}$ is linearly dependent, then $a \cdot u + b \cdot v = 0$, where $a \neq 0$ or $b \neq 0$

$\Rightarrow a \cdot u = -b \cdot v$

$\Rightarrow$ If $a \neq 0$, then $u = (-b/a) \cdot v$. Otherwise, $b \neq 0$, then $v = (-a/b) \cdot u$

$\Rightarrow u$ or v is a multiple of the other

$(\Leftarrow)$

Suppose u or v is a multiple of the other, then $u = c \cdot v$ or $v = d \cdot u$

$\Rightarrow u - c \cdot v = 0$ or $d \cdot u - v = 0$

For $a \cdot u + b \cdot v = 0$, we can find at least one nontrivial solution $(a, b) = (1, -c)$ or $(d, -1)$

$\Rightarrow \{u, v\}$is linearly dependent

Q.E.D.

Sec. 1.6

1.6.2. Determine which of the following sets are bases for R^3.

(c) {(1, 2, -1), (1, 0, 2), (2, 1, 1)}

Ans.:

$\because$The only solution of a(1, 2, -1)+b(1, 0, 2)+c(2, 1, 1)=0 is $\{a, b, c\}=\{0, 0, 0\}$

$\therefore$ {(1, 2, -1), (1, 0, 2), (2, 1, 1)} is linearly independent

$\because \dim(R^3) = 3$ = the number of vectors in the set {(1, 2, -1), (1, 0, 2), (2, 1, 1)}

According to Corollary 2 of Theorem 1.10, {(1, 2, -1), (1, 0, 2), (2, 1, 1)} is a basis of R^3.

1.6.3. Determine which of the following sets are bases for $P_2(R)$.

(c) $\{1-2x-2x^2, -2+3x-x^2, 1-x+6x^2\}$

Ans.:

$\because$ The only solution of $a(1-2x-2x^2)+b(-2+3x-x^2)+c(1-x+6x^2)=0$ is $a=b=c=0$

∴ $\{1-2x-2x^2, -2+3x-x^2, 1-x+6x^2\}$ is linearly independent.
∵ $\dim(P_2(R)) = 3$ = the number of vectors in the set $\{1-2x-2x^2, -2+3x-x^2, 1-x+6x^2\}$
According to Corollary 2 of Theorem 1.10, $\{1-2x-2x^2, -2+3x-x^2, 1-x+6x^2\}$ is basis for $P_2(R)$.

1.6.12. Let u, v, and w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u+v+w, v+w, w\}$ is also a basis for V.

Ans.:

If $\{u, v, w\}$ is a basis for V, then the only solution for $au+bv+cw=0$ is $a=b=c=0$
Now, we want to find the solution of $a_1(u+v+w)+b_1(v+w)+c_1w = 0$
$\Rightarrow a_1u+ a_1v+ a_1w+ b_1v+ b_1w+ c_1w = 0$
$\Rightarrow a_1u+(a_1+ b_1) v+(a_1+ b_1+ c_1) w = 0$
the only one solution is $a_1= a_1+ b_1= a_1+ b_1+ c_1=0$
$\Rightarrow a_1= b_1= c_1=0$
∴ the only one solution of $a_1(u+v+w)+b_1(v+w)+c_1w = 0$ is $a_1= b_1= c_1=0$
$\Rightarrow$ $\{u+v+w, v+w, w\}$ is linearly independent
∵ $\dim(P_2(R)) = 3$ = the number of vectors in the set $\{u+v+w, v+w, w\}$
According to Corollary 2 of Theorem 1.10, $\{u+v+w, v+w, w\}$ is also a basis for V.
Q.E.D.

1.6.14. Find bases for the following subspaces of F^5:

$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5: a_1-a_3-a_4=0\}$

and

$W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5: a_2 = a_3 = a_4 \text{ and } a_1+a_5 = 0\}$.

What are the dimensions of W_1 and W_2 ?

Ans.:

(A)

For any $v \in W_1$, $v = (a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_3, a_1 - a_3, a_5)$
$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + (a_1 - a_3)(0,0,0,1,0) + a_5(0,0,0,0,1)$
$= a_1(1,0,0,1,0) + a_2(0,1,0,0,0) + a_3(0,0,1,-1,0) + a_5(0,0,0,0,1)$
∵ $\{(1,0,0,1,0), (0,1,0,0,0), (0,0,1,-1,0), (0,0,0,0,1)\}$ is linearly independent.
∴ The basis set for W_1 is $\{(1,0,0,1,0), (0,1,0,0,0), (0,0,1,-1,0), (0,0,0,0,1)\}$, $\dim(W_1)=4$

(B)

For any $v \in W_2$, $v = (a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_2, a_2, -a_1)$
$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_2(0,0,1,0,0) + a_2(0,0,0,1,0) - a_1(0,0,0,0,1)$
$= a_1(1,0,0,0,-1) + a_2(0,1,1,1,0)$
∵ $\{(1,0,0,0,-1), (0,1,1,1,0)\}$ is linearly independent.

$\therefore$ The basis set for W_2 is {(1,0,0,0,-1), (0,1,1,1,0)}, $\dim(W_2)=2$

1.6.31. Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geqq n$.

(a) Prove that $\dim(W_1 \cap W_2) \le n$.

(b) Prove that $\dim(W_1 + W_2) \leqq m + n$.

Ans.:

(a)

According to Theorem 1.11:

$\because W_1 \cap W_2$ is a subspace of W_1 and W_2 ,

$\therefore \dim(W_1 \cap W_2) \le \dim(W_2) = n$

Q.E.D

(b)

$\because \dim(W_1) = m,\ \dim(W_2) = n,$

Suppose the basis for W_1 is $\{x_1, x_2, ..., x_m\}$ and the basis for W_2 is $\{y_1, y_2, ..., y_n\}$

(i)

For any p in $W_1 + W_2$, $p = p_1 + p_2$, where $p_1 \in W_1$ and $p_2 \in W_2$,

then $p = (a_1x_1 + a_2x_2 + ... + a_mx_m) + (b_1y_1 + b_2y_2 + ... + b_ny_n)$

$= (a_1x_1 + a_2x_2 + ... + a_mx_m + b_1y_1 + b_2y_2 + ... + b_ny_n)$

$\therefore p \in \mathrm{Span}(\{x_1, x_2, ..., x_m, y_1, y_2, ..., y_n\})$

$\Rightarrow W_1 + W_2 \subseteq \mathrm{Span}(\{x_1, x_2, ..., x_m, y_1, y_2, ..., y_n\})$

(ii)

For any $q \in \mathrm{Span}(\{x_1, x_2, ..., x_m, y_1, y_2, ..., y_n\})$,

then $q = a_1x_1 + a_2x_2 + ... + a_mx_m + b_1y_1 + b_2y_2 + ... + b_ny_n$

$= (a_1x_1 + a_2x_2 + ... + a_mx_m) + (b_1y_1 + b_2y_2 + ... + b_ny_n)$

$= q_1 + q_2 \in W_1 + W_2$ where $q_1 \in W_1$ and $q_2 \in W_2$

$\mathrm{Span}(\{x_1, x_2, ..., x_m, y_1, y_2, ..., y_n\}) \subseteq W_1 + W_2$

Base on (i) and (ii), $W_1 + W_2 = \mathrm{Span}(\{x_1, x_2, ..., x_m, y_1, y_2, ..., y_n\})$

$\Rightarrow \dim(W_1 + W_2) = \dim(\mathrm{Span}(\{x_1, x_2, ..., x_m, y_1, y_2, ..., y_n\})) \le m + n$

Q.E.D.