Solution 3

Sec2.1

2.1.10

Suppose that T: $R_2 \to R_2$ is linear, T(1,0) = (1,4), and T(1,1) = (2,5). What is T(2,3)? Is T one-to-one?

Ans.:

(A)

(2,3) = (-1)(1,0)+3(1,1)

$\therefore$ T(2,3) = (-1)T(1,0)+3T(1,1) = (-1)(1,4)+3(2,5) = (5,11)

(B)

$\because$ The two vectors (1, 0) and (1, 1) are linearly independent.

$\Rightarrow$ For any $x \in R_2$, x has the form x=a(1,0)+b(1,1).

Suppose that T(x) = 0

$\Rightarrow$ T(x)=aT(1,0)+bT(1,1)=0

$\Rightarrow$ a(1,4)+b(2,5)=0

$\Rightarrow$ (a+2b,4a+5b)=0

$\Rightarrow$ (a,b)=(0,0)

The only solution for T(x)=0 is x=(0,0).

$\Rightarrow$ N(T)={0}

$\therefore$ T is one-to-one.

2.1.16

Let T: $P(R) \to P(R)$ be defined by $T(f(x)) = f'(x)$. Recall that T is linear. Prove that T is onto, but not one-to-one.

Ans.:

$\because$ $T(1) = T(2) = 0$

$\therefore$ $N(T) \neq \{0\}$

$\Rightarrow$ T is not one-to-one.

For any element $g(x)$ in $P(R)$, we can find $f(x) = \int_c g(x)dx$ in $P(R)$ which satisfies $T(f(x)) = f'(x) = g(x)$

$\therefore$ $g(x)$ is in R(T).

$\therefore$ $P(R) \subseteq R(T)$

$\because$ $R(T) \subseteq P(R)$

$\therefore$ $P(R) = R(T)$

$\Rightarrow$ T is onto.

2.1.21

Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions T, U: $\mathrm{V} \to \mathrm{V}$ by

$$\mathrm{T}(a_1, a_2, ...) = (a_2, a_3 ...) \quad \text{and} \quad \mathrm{U}(a_1, a_2, ...) = (0, a_1, a_2 ...) .$$

T and U are called the **left shift** and **right shift** operators on V, respectively.

(a) Prove that T and U are linear.

(b) Prove that T is onto, but not one-to-one.

(c) Prove that U is one-to-one, but not onto.

Ans.:

(a)

Let $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ denote any two vectors in V and c denote a scalar.

(i) $\mathrm{T}(\mathrm{c}x + y) = \mathrm{T}(\mathrm{c}x_1 + y_1, \mathrm{c}x_2 + y_2, ...) = (\mathrm{c}x_2 + y_2, \mathrm{c}x_3 + y_3, ...) = (\mathrm{c}x_2, \mathrm{c}x_3, ...) + (y_2, y_3, ...)$

$= c(x_2, x_3, ...) + (y_2, y_3, ...) = \mathrm{cT}(x_1, x_2, ...) + \mathrm{T}(y_1, y_2, ...) = c\mathrm{T}(x) + \mathrm{T}(y)$

$\therefore$T is linear.

(ii)Similarly,

$\mathrm{U}(\mathrm{c}x + y) = \mathrm{U}(\mathrm{c}x_1 + y_1, \mathrm{c}x_2 + y_2, ...) = (0, \mathrm{c}x_1 + y_1, \mathrm{c}x_2 + y_2, ...) = (0, \mathrm{c}x_1, \mathrm{c}x_2, ...) + (0, y_1, y_2, ...)$

$= c(0, x_1, x_2, ...) + (0, y_1, y_2, ...) = \mathrm{cU}(x_1, x_2, ...) + \mathrm{U}(y_1, y_2, ...) = c\mathrm{U}(x) + \mathrm{U}(y)$

$\therefore$U is linear, too.

(b)

$\because \mathrm{T}(1, a_2, ...) = \mathrm{T}(2, a_2, ...) = (a_2, a_3 ...)$

$\therefore$T is not one-to-one.

For any vector $x = (x_1, x_2, ...)$ in V , we can find $y = (1, x_1, x_2, ...)$ in V which satisfies $\mathrm{T}(y) = x$

$\Rightarrow x \in \mathrm{R(T)}$

$\therefore \mathrm{V} \subseteq \mathrm{R(T)}$

$\because \mathrm{R(T)} \subseteq \mathrm{V}$

$\therefore \mathrm{R(T)} = \mathrm{V}$

$\Rightarrow$T is onto.

(c) $\mathrm{U}(a_1, a_2, ...) = (0, a_1, a_2 ...)$

Clearly, if $\mathrm{U}(a_1, a_2, ...) = (0,0,0...)$, then $a_1 = a_2 = ... = 0$

$\therefore$N(U)={0}

$\Rightarrow$U is one-to-one.

$\because \mathrm{U}(a_1, a_2, ...) = (0, a_1, a_2 ...)$

$\Rightarrow$There are not solution for function $\mathrm{U}(a_1, a_2, ...) = (2, a_1, a_2 ...)$.

$\therefore$U is not onto.

2.1.35

Assume the definition of *direct sum* given in the exercises of Section 1.3. Let V be a finite-dimensional vector space and T: $\mathrm{V} \to \mathrm{V}$ be linear.

(a) Suppose that V = R(T) + N(T). Prove that V=R(T)$\oplus$N(T).

(b) Suppose that R(T)$\cap$N(T) = {*0*}. Prove that V=R(T)$\oplus$N(T).

Be careful to say in each part where finite-dimensionality is used.

Ans.:

(a)

Let the dimension of V be *n*.

Suppose that $\nu = \{v_1, v_2, ..., v_m\}$ is a basis for N(T), according to Corollary of Theorem1.11 in Sec. 1.6, by extending ν, we have $\alpha = \{v_1, v_2, ..., v_m, v_{m+1}, ..., v_n\}$ which is a basis for V.

For any vector *u* in R(T),

$$u = \sum_{i=1}^{n} a_i \mathrm{T}(v_i) = \sum_{i=1}^{m} a_i \mathrm{T}(v_i) + \sum_{i=m+1}^{n} a_i \mathrm{T}(v_i) = \sum_{i=m+1}^{n} a_i \mathrm{T}(v_i)$$

$\Rightarrow \{\mathrm{T}(v_{m+1}), ..., \mathrm{T}(v_n)\}$ generates R(T)

Let $\beta = \{v_1, v_2, ..., v_m, \mathrm{T}(v_{m+1}), ..., \mathrm{T}(v_n)\}$

$\because$ V = R(T) + N(T)

$\therefore$For any vector $w = w_1 + w_2 \in \mathrm{V}$ where $w_1 \in \mathrm{R}(T)$ and $w_2 \in \mathrm{N}(T)$

$$\Rightarrow w = \sum_{i=1}^{m} a_i v_i + \sum_{i=m+1}^{n} a_i \mathrm{T}(v_i)$$

$\therefore \beta = \{v_1, v_2, ..., v_m, \mathrm{T}(v_{m+1}), ..., \mathrm{T}(v_n)\}$ generates V

Because the numbers of vectors in β is *n*=dim(V).

According to Corollary 2 of Theorem 1.10, $\beta = \{v_1, v_2, ..., v_m, \mathrm{T}(v_{m+1}), ..., \mathrm{T}(v_n)\}$ is also a basis for V.

Let *x* be a vector in R(T)$\cap$N(T).

$\because x \in \mathrm{N(T)} \therefore x = a_1 v_1 + a_2 v_2 + ... + a_m v_m$

$\because x \in \mathrm{R(T)} \therefore x = a_{m+1}\mathrm{T}(v_{m+1}) + a_{m+2}\mathrm{T}(v_{m+2}) + ... + a_n \mathrm{T}(v_n)$

$\Rightarrow a_1 v_1 + a_2 v_2 + ... + a_m v_m = a_{m+1}\mathrm{T}(v_{m+1}) + a_{m+2}\mathrm{T}(v_{m+2}) + ... + a_n \mathrm{T}(v_n)$

$\Rightarrow a_1 v_1 + a_2 v_2 + ... + a_m v_m - a_{m+1}\mathrm{T}(v_{m+1}) - a_{m+2}\mathrm{T}(v_{m+2}) - ... - a_n \mathrm{T}(v_n) = 0$

$\because \beta = \{v_1, v_2, ..., v_m, \mathrm{T}(v_{m+1}), ..., \mathrm{T}(v_n)\}$ is linearly independent.

$\therefore$The only solution is $a_1 = a_2 = ... = a_m = a_{m+1} = a_{m+2} = ... = a_n = 0 \Rightarrow x=0$

$\Rightarrow$R(T)$\cap$N(T) = {*0*}

According to V = R(T)+N(T) and R(T)$\cap$N(T) = {*0*}, we have V = R(T)$\oplus$N(T)

(b)

dim(R(T)+N(T))=dim(R(T))+dim(N(T))-dim(R(T)$\cap$N(T))

Suppose that R(T)$\cap$N(T)={0}, then dim(R(T)$\cap$N(T))=0.

$\Rightarrow$dim(R(T)+N(T))=dim(R(T))+dim(N(T))=dim(V) (Dimension Theorem)

Then, according to Sec. 1.3 exercises 23(a), R(T)+N(T) is a subspace of V.

$\therefore$ R(T)+N(T)=V

According to R(T)+N(T)=V and R(T)$\cap$N(T)={0}, we have V=R(T)$\oplus$N(T).

Sec2.2

2.2.2

Let β and γ be the standard ordered bases for R^n and R^m, respectively. For each linear transformation T: $R^n \to R^m$, compute $[T]_\beta^\gamma$.

(b) T: $R^2 \to R^3$ defined by $T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$.

Ans.:

(b)

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

2.2.4

Define

$$T: M_{2x2}(R) \to P_2(R) \text{ by } T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \right\} \text{ and } \gamma = \{1, x, x^2\}.$$

Compute $[T]_\beta^\gamma$.

Ans.:

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 + 0x + 0x^2$, $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + 0x + 1x^2$,

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 + 0x + 0x^2$, $T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 + 2x + 0x^2$

$$\therefore [T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

2.2.8

Let V be an *n*-dimensional vector space with an ordered basis β. Define T: $\mathrm{V} \to \mathrm{F}^n$ by $\mathrm{T}(x) = [x]_\beta$. Prove that T is linear.

Ans.:

Let $\beta = \{x_1, x_2, ..., x_n\}$.

For any $y = a_1x_1 + a_2x_2 + ... + a_nx_n$ and $z = b_1z_1 + b_2z_2 + ... + b_nz_n$ in V, we have

$$\mathrm{T}(y) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad \mathrm{T}(z) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Let $c \in \mathrm{F}$

$$\Rightarrow \mathrm{T}(cy + z) = T((ca_1 + b_1)x_1 + ... + (ca_n + b_n)x_n) = \begin{pmatrix} ca_1 + b_1 \\ \vdots \\ ca_n + b_n \end{pmatrix}$$

$$= c\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = c\mathrm{T}(y) + T(z)$$

$\therefore$ T is linear.