Solution 4

Sec2.3

2.3.2(a)

Let

$$A=\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix},\quad B=\begin{pmatrix}1 & 0 & -3\\ 4 & 1 & 2\end{pmatrix},\quad C=\begin{pmatrix}1 & 1 & 4\\ -1 & -2 & 0\end{pmatrix},\ \text{and}\quad D=\begin{pmatrix}2\\ -2\\ 3\end{pmatrix}$$

Compute *A*(2*B*+3*C*), (*AB*)*D*, and *A*(*BD*).

Ans.:

$$2B+3C=\begin{pmatrix}2 & 0 & -6\\ 8 & 2 & 4\end{pmatrix}+\begin{pmatrix}3 & 3 & 12\\ -3 & -6 & 0\end{pmatrix}=\begin{pmatrix}5 & 3 & 6\\ 5 & -4 & 4\end{pmatrix}$$

$$A(2B+3C)=\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix}\begin{pmatrix}5 & 3 & 6\\ 5 & -4 & 4\end{pmatrix}=\begin{pmatrix}20 & -9 & 18\\ 5 & 10 & 8\end{pmatrix}$$

$$AB=\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & -3\\ 4 & 1 & 2\end{pmatrix}=\begin{pmatrix}13 & 3 & 3\\ -2 & -1 & -8\end{pmatrix}$$

$$(AB)D=\begin{pmatrix}13 & 3 & 3\\ -2 & -1 & -8\end{pmatrix}\begin{pmatrix}2\\ -2\\ 3\end{pmatrix}=\begin{pmatrix}29\\ -26\end{pmatrix}$$

$$BD=\begin{pmatrix}1 & 0 & -3\\ 4 & 1 & 2\end{pmatrix}\begin{pmatrix}2\\ -2\\ 3\end{pmatrix}=\begin{pmatrix}-7\\ 12\end{pmatrix}$$

$$A(BD)=\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix}\begin{pmatrix}-7\\ 12\end{pmatrix}=\begin{pmatrix}29\\ -26\end{pmatrix}$$

2.3.13

Let *A* and *B* be *n*x*n* matrices. Recall that the trace of *A* is defined

$$\mathrm{tr}(A)=\sum_{i=1}^{n}A_{ii}$$

Prove that tr(*AB*)=tr(*BA*) and tr(*A*)=tr(A^t).

Ans.:.

(i)

$$(AB)_{ii}=\sum_{k=1}^{n}A_{ik}B_{ki}$$

$(BA)_{ii} = \sum_{k=1}^{n} B_{ik} A_{ki}$

tr(AB)= $\sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik} B_{ki}$

tr(BA)=

$$\sum_{i=1}^{n}\sum_{k=1}^{n} B_{ik} A_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ki} B_{ik} \overset{\text{Let } k=i \text{ and } i=k}{=} \sum_{k=1}^{n}\sum_{i=1}^{n} A_{ik} B_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik} B_{ki}$$

= tr(AB)

(ii)

$\because (A^t)_{ii} = A_{ii}$

$\therefore$ tr(A) = $\sum_{i=1}^{n} A_{ii}$ = tr(A^t)

Sec2.4

2.4.2

For each of the following linear transformations T, determine whether T is invertible and justify your answer.

(c) T: $R^3 \to R^3$ defined by T(a_1,a_2,a_3) = ($3a_1$-$2a_3$, a_2, $3a_1$+$4a_2$).

Ans.:

According to the definition of T,

$$\Rightarrow \text{T: } \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 3a_1 - 2a_3 \\ a_2 \\ 3a_1 + 4a_2 \end{pmatrix}$$

We can find a transformation T^{-1} from $R^3 \to R^3$ which satisfies

$$T^{-1}(3a_1-2a_3, a_2, 3a_1+4a_2) = (a_1, a_2, a_3) \left(\begin{pmatrix} 0 & -4/3 & 1/3 \\ 0 & 1 & 0 \\ -1/2 & -2 & 1/2 \end{pmatrix}\begin{pmatrix} 3a_1 - 2a_3 \\ a_2 \\ 3a_1 + 4a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right)$$

$\therefore$T is invertible.

2.4.4

Let A and B be nxn invertible matrices. Prove that AB is invertible and $(AB)^{-1}=B^{-1}A^{-1}$.

Ans.:

$\because (AB)(B^{-1}A^{-1})=A(B(B^{-1}A^{-1})=A((BB^{-1})A^{-1})=A(IA^{-1})=AA^{-1}=I$

$(B^{-1}A^{-1})\ (AB)=B^{-1}(A^{-1}(AB))=B^{-1}(A^{-1}A)B)=B^{-1}B=I$

$\therefore AB$ is invertible and $(AB)^{-1}=B^{-1}A^{-1}$.

2.4.14

Let

$$V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}$$

Construct an isomorphism from V to F^3.

Ans.:

We can find $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis for V then

$$\begin{pmatrix} \mathrm{a} & a+b \\ 0 & c \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We define that T is a linear transformation from V to F^3 (T: V→F^3), where

$$T\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right) = (1,0,0), \quad T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = (0,1,0), \quad T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) = (0,0,1)$$

$$\therefore T\left(\begin{pmatrix} \mathrm{a} & a+b \\ 0 & c \end{pmatrix} \right) = (a, b, c)$$

$\Rightarrow$ T is invertible.

$\Rightarrow$ T is an isomorphism from V to F^3.

2.4.17

Let V and W be finite-dimensional vector spaces and T: V→W be an isomorphism.

Let V_0 be a subspace of V.

(a) Prove that $T(V_0)$ is a subspace of W.

(b) Prove that $\dim(V_0)=\dim(T(V_0))$.

Ans.:

(a) we define that 0_v is the zero vector in the vector space V.

(i) $\because V_0$ is a subspace of V.

$\therefore 0_v \in V_0$

$\because$ T is linear.

$\therefore T(0_v) = 0_w \Rightarrow 0_w \in T(V_0)$

(ii) Let any two vectors x_1, x_2 in V_0 and $T(x_1)=y_1$, $T(x_2)=y_2 \in T(V_0)$.

Then let $a \in F$

$\because V_0$ is a vector space.

$\therefore ax_1 + x_2 \in V_0$

$\Rightarrow \mathrm{T}(ax_1 + x_2) \in \mathrm{T}(\mathrm{V}_0)$
$\Rightarrow \mathrm{T}(ax_1) + \mathrm{T}(x_2) \in \mathrm{T}(\mathrm{V}_0)$
$\Rightarrow a\mathrm{T}(x_1) + \mathrm{T}(x_2) \in \mathrm{T}(\mathrm{V}_0)$
$\Rightarrow ay_1 + y_2 \in \mathrm{T}(\mathrm{V}_0)$

According to (i) and (ii), T(V_0) is a subspace of W.

(b) Let $\beta_0 = \{x_1, x_2, ..., x_m\}$ is a basis of V_0 and dim(V_0)=m.

We define $\gamma_0 = \{\mathrm{T}(x_1), \mathrm{T}(x_2), ..., \mathrm{T}(x_m)\}$.

∵T is isomorphism

∴T is linear and one-to-one.

Suppose that $\gamma_0 = \{\mathrm{T}(x_1), \mathrm{T}(x_2), ..., \mathrm{T}(x_m)\}$ is linearly dependent, then

$a_1\mathrm{T}(x_1) + a_2\mathrm{T}(x_2) + ... + a_m\mathrm{T}(x_m) = 0$ has non-trivial solution.

∵T is linear.

∴ $a_1x_1 + a_2x_2 + ... + a_mx_m = 0$ has non-trivial solution, too.

$\Rightarrow \beta_0 = \{x_1, x_2, ..., x_m\}$ is linearly dependent. $\Rightarrow$ conflicting

∴ $\gamma_0 = \{\mathrm{T}(x_1), \mathrm{T}(x_2), ..., \mathrm{T}(x_m)\}$ is linearly independent.

According to Theorem 2.2, T(V_0)=span(γ_0).

∵ γ_0 is linearly independent and the number of the vectors in γ_0 is m.

∴dim(T(V_0))=m.

dim(V_0)=dim(T(V_0))=m.

Q.E.D.