Solution 6

Sec4.1

4.1.3

Compute the determinants of the following matrices in $M_{2x2}(C)$

(b) $\begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix}$

Ans.:

36+41*i*

4.1.4

For each of the following pairs of vectors *u* and *v* in R^2, compute the area of the parallelogram determined by *u* and *v*.

(a) *u*=(3, -2) and *v*=(2, 5)

Ans.:

$$\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right| = \left|\det\begin{pmatrix} 3 & -2 \\ 2 & 5 \end{pmatrix}\right| = |19| = 19$$

Sec4.2

4.2.3

Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Ans.:

$$\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\because \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times 2 = \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \quad \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times 3 = \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

$$\begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \overset{(5)}{\leftarrow} = \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

$$\begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times 7 = \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{vmatrix} =$$

$$\det\begin{pmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{pmatrix}$$

$\therefore$k= $2\times3\times7$ =42

4.2.7

Evaluate the determinant of the given matrix by cofactor expansion along the indicated row.

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$$

Ans.:

$\det(A)=(-1)^{2+1} \cdot (-1) \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + (-1)^{2+2} \cdot 0 \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} + (-1)^{2+3} \cdot (-3) \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = -6+3(-2)=-12$

4.2.22

$$\begin{pmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{pmatrix}$$

Ans.:

$$\det\begin{vmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & -12 \\ 0 & 2 & 1 & -41 \\ 0 & 4 & 7 & -77 \\ 0 & 1 & -2 & -33 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 2 & 1 & -41 \\ 4 & 7 & -77 \\ 1 & -2 & -33 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 5 & 25 \\ 0 & 15 & 55 \\ 1 & -2 & -33 \end{vmatrix} = (-1)^{3+1} \cdot 1 \cdot \begin{vmatrix} 5 & 25 \\ 15 & 55 \end{vmatrix} = -100$$

Sec4.3

4.3.5

Use Cramer's rule to solve the given system of linear equations.

$x_1 - x_2 + 4x_3 = -4$

$-8x_1 + 3x_2 + x_3 = 8$

$2x_1 - x_2 + x_3 = 0$

Ans.:

$$\det\begin{pmatrix} 1 & -1 & 4 \\ -8 & 3 & 1 \\ 2 & -1 & 1 \end{pmatrix} = 2$$

$$x_1 = \frac{1}{2}\begin{pmatrix} -4 & -1 & 4 \\ 8 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix} = -20$$

$$x_2 = \frac{1}{2}\begin{pmatrix} 1 & -4 & 4 \\ -8 & 8 & 1 \\ 2 & 0 & 1 \end{pmatrix} = -48$$

$$x_3 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -4 \\ -8 & 3 & 8 \\ 2 & -1 & 0 \end{pmatrix} = -8$$

Sec5.1

5.1.2

For each of the following linear operators T on a vector space V and ordered bases β,

compute $[\mathrm{T}]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.

(c) V=R^3, $\mathrm{T}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a+2b-2c \\ -4a-3b+2c \\ -c \end{pmatrix}$, and $\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$

Ans.:

$$\det(\mathrm{T}-\lambda\mathrm{I}) = \begin{pmatrix} 3-\lambda & 2 & -2 \\ -4 & -3-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (1-\lambda^2)(1+\lambda) = (1+\lambda)^2(1-\lambda)$$

$$\Rightarrow \lambda = 1,-1,-1$$

$$\Rightarrow [\mathrm{T}]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\lambda = 1 \Rightarrow (\mathrm{T}-\lambda\mathrm{I}) = \begin{pmatrix} 2 & 2 & -2 \\ -4 & -4 & 2 \\ 0 & 0 & -2 \end{pmatrix} \Rightarrow \text{eigenvector=}\left\{ t\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},\ \ t \in R \right\}$$

$$\lambda = -1 \Rightarrow (\mathrm{T}-\lambda\mathrm{I}) = \begin{pmatrix} 4 & 2 & -2 \\ -4 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{eigenvector=}\left\{ t_1\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ t_1, t_2 \in R \right\}$$

$$\Rightarrow B = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \right\} \text{ is a basis consisting of eigenvectors of T.}$$

5.1.3

For each of the following matrices $A \in \mathrm{M}_{nxn}(F)$,

(i) Determine all the eigenvalues of *A*.

(ii) For each eigenvalue λ of *A*, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for F^n consisting of eigenvectors of *A*.

(iv) If successful in finding such a basis, determine an invertible matrix *Q* and a diagonal matrix *D* such that $Q^{-1}AQ = D$

(b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = R$

Ans.:

(b)

(i) $\det(A-\lambda I) = 0 \Rightarrow \det\begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix} = 0$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0 \Rightarrow \lambda = 1,2,3$$

(ii) $\lambda = 1 \Rightarrow \begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \left\{ t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, t \in R \right\}$

$\lambda = 2 \Rightarrow \begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \left\{ t\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, t \in R \right\}$

$\lambda = 3 \Rightarrow \begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \left\{ t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, t \in R \right\}$

(iii) $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

(iv) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}, Q^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -2 \end{pmatrix}$

$$QAQ^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

5.1.4

For each linear operator T on V, find the eigenvalues of T and an ordered basis β of V such that $[\mathrm{T}]_\beta$ is a diagonal matrix.

(d) V=P_1(*R*) and T(*ax*+*b*) = (-6*a*+2*b*)*x*+(-6*a*+*b*)

Ans.:

a=0, *b*=1 $\Rightarrow$ T(1)=2*x*+1

a=1, *b*=0 $\Rightarrow$ T(*x*)=-6*x*-6

$\Rightarrow \mathrm{A} = \begin{pmatrix} 1 & -6 \\ 2 & -6 \end{pmatrix}$

$\det(\mathrm{A} - \lambda \mathrm{I}) = \det\begin{pmatrix} 1-\lambda & -6 \\ 2 & -6-\lambda \end{pmatrix} = (\lambda - 1)(\lambda + 6) + 12$

$\Rightarrow \lambda = -2, -3$

$\lambda = -2 \Rightarrow \mathrm{A} - \lambda\mathrm{I} = \begin{pmatrix} 3 & -6 \\ 2 & -4 \end{pmatrix}$, eigenvector= $\left\{ t\begin{pmatrix} 2 \\ 3 \end{pmatrix}, t \in R \right\}$

$\lambda = -3 \Rightarrow \mathrm{A} - \lambda\mathrm{I} = \begin{pmatrix} 4 & -6 \\ 2 & -3 \end{pmatrix}$, eigenvector= $\left\{ t\begin{pmatrix} 2 \\ 1 \end{pmatrix}, t \in R \right\}$

$[\mathrm{T}]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$

5.1.9

Prove that the eigenvalues of an upper triangular matrix *M* are the diagonal entries of *M*.

Ans.:

Theorem 4.11. If *A* is a triangular *n*x*n* matrix, then det(*A*) =$A_{11}A_{22}\ldots A$nn; that is, the determinant of *A* is the product of the entries of *A* that lie on the diagonal.

PROOF. Let *A* be an upper triangular *n*x*n* matrix. The proof is by induction on *n*. If *n*=2, then *A* has the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

And so det(*A*) =$A_{11}A_{22}$-A_{12} · 0=$A_{11}A_{22}$, proving the theorem for upper triangular matrices if *n*=2.

Assume that the theorem is true for upper triangular *n*x*n* matrix. Then *A* has the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1(n-1)} & A_{1n} \\ 0 & A_{22} & \cdots & A_{2(n01)} & A_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & A_{nn} \end{pmatrix}$$

Expanding along the first column, we see that

$$\det(A) = A_{11} \cdot \det\begin{pmatrix} A_{22} & \cdots & A_{2(n-1)} & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & A_{nn} \end{pmatrix} = A_{11} \cdot (A_{22}\ldots A_{nn})$$

by the induction hypothesis. This completes the induction and proves the theorem for upper triangular matrices.

If *A* is a lower triangular matrix, then A^t is an upper triangular matrix. Hence the first part of this proof and Theorem 4.10 imply that

$\det(A)=\det(A^t)=(A^t)_{11}\ldots(A^t)_{nn}.=A_{11}\ldots A_{nn}$.

5.1.12

(a) Prove that similar matrices have the same characteristic polynomial.

(b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.

Ans.:

(a) A, B is similar $\rightarrow A=Q^{-1}BQ$

$\det(A-tI)$

$= \det(Q^{-1}BQ-t\,Q^{-1}Q)$

$= \det(Q^{-1}BQ- Q^{-1}(tI)Q)$

$= \det(Q^{-1}(BQ-(tI)Q))$

$= \det(Q^{-1})\det(BQ-(tI)Q)$

$= \det(Q^{-1})\det((B-tI)Q)$

$= \det(Q^{-1})\det(B-tI)\det(Q)$

$= \det(Q^{-1}Q)\det(B-tI)$

$= \det(B-tI)$

$\therefore \det(A-tI) = \det(B-tI)$

$\therefore$ Similar matrices have the same characteristic polynomial.

Q.E.D

(b) According to Theorem 2.23,

Let T be a linear operator on a finite-dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β' - coordinates into β - coordinates.

Then

$$[\mathrm{T}]_{\beta'} = Q^{-1}[\mathrm{T}]_{\beta}Q$$

$\therefore$ Following (a), the linear operator $[\mathrm{T}]_{\beta'}$ and $[\mathrm{T}]_{\beta}$ have the same characteristic polynomial.