Solution 7

Sec. 5.2

2.

For the matrix $A \in M_{nxn}(R)$, test A for diagonal and ability and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ=D$

(d) $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$

Ans.:

(d) $(3-\lambda)[(-5-\lambda)+32]=0 \Rightarrow \lambda = 3,3,-1$

$$(A-3I)=\begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \Rightarrow \text{eigenvector} = (1,-1,0) \text{ and } (0,0,1)$$

$$(A+I)=\begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 4 \end{pmatrix} \Rightarrow \text{eigenvector} = (1,2,0)$$

$$Q=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

3.

(c) For the linear operators T on a vector space V, test T for diagonalizability and if T is a diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

Ans.:

(c) $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \Rightarrow [T]_\gamma=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$\det([T]_\gamma-1)=\begin{pmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}=0 \Rightarrow (\lambda-2)(\lambda^2-1)=0 \Rightarrow \text{P269 condition 1 fails}$$

to hold $\Rightarrow$ T is not a diagonalizable matrix.

12.

Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1}. Prove that the eigenvalue of T corresponding to λ is the same as the eigenspace T^{-1} of corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Ans.:

(a) For each eigenvector ν and correspond eigenvalue λ of linear operator T.

By definition, we have T(ν)=$\lambda\nu$.

Therefore, $\nu = T^{-1}(T(\nu)) = T^{-1}(\lambda\nu) \Rightarrow T^{-1}(\nu) = \frac{1}{\lambda}\nu$

$\therefore$ ν is also the eigenvector and $\frac{1}{\lambda}$ si correspond eigenvalue of linear operator T^{-1}

$\therefore$The eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} correspond to $\frac{1}{\lambda}$.

Q.E.D.

(b) Assume T is a linear operator on V, a finite-dimensional vector space of dimension *n*.

$\because$T is diagonalizable.

$\therefore$T has *n* independent eigenvectors.

By (a), any eigenvector of T is also the eigenvector of T^{-1}.

$\therefore$ T^{-1} also has *n* independent eigenvectors.

$\therefore$ T^{-1} is diagonalizable.

Q.E.D.

Sec. 5.4

2.

(e) For the linear operator T on the vector space V, determine whether the given subspace W is a T-invariant subspace of V.

$$V = M_{2x2}(R), T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A \;\&\; W = \{A \in V : A^t = A\}$$

Ans.:

$$T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix} \neq A^t \Rightarrow \text{W is not T-invariant.}$$

6.

(c) For the linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector *z*. $V = M_{2x2}(R), T(A) = A^t$ & $Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Ans.:

$$T(Z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = Z \Rightarrow \text{Basis is Z.}$$

36.

Let T be a linear operator on a finite-dimensional vector space V. Prove that T is diagonalizable iff V is the direct sum of one-dimensional T-invariant subspaces.

Ans.:

$(\Rightarrow)$

Let dim(V)=n

If T is diagonalizable, there exists a basis $\beta = \{v_1, v_2, ..., v_n,\}$ consisting of eigenvectors of T. Let $W_i = span\{v_i\}$ for $1 \le i \le n$

$\forall w \in W_i$, we can be represented as $w = av_i$ for some a,

$T(w) = T(av_i) = aT(v_i) = a\lambda_i v_i \in W_i$, where λ_i is the eigenvalue corresponding to v_i.

$\therefore W_i$ is a 1-dimensional T-invariant subspace of V.

$\forall w = \sum_{i=1}^{n} a_i v_i \in V$, let $w_i = a_i v_i \in W_i$

Then $w = w_1 + w_2 + ... + w_n \in W_1 + W_2 + ... + W_n$

$\therefore V \subseteq W_1 + W_2 + ... + W_n$

$\forall w = w_1 + w_2 + ... + w_n \in W_1 + W_2 + ... + W$ where $w_i \in W$

Representing $w_i = b_i v_i$ for $1 \le i \le n$,

We have $w = \sum_{i=1}^{n} b_i v_i \in V$

$\therefore W_1 + W_2 + ... + W_n \subseteq V$

Let $w \in W_j \cap \sum_{i \ne j} W_i$

$$w \in W_j \Rightarrow w = a_j v_j$$

$$w \in \sum_{i \ne j} V_i \Rightarrow w = \sum_{i \ne j} a_i v_i$$

$$a_j v_j = \sum_{i \ne j} a_i v_i \Rightarrow \sum_{i \ne j} a_i v_i + (-a_j) v_j = 0$$

But $\{v_1, v_2, ..., v_n\}$ are independent.
$\therefore a_1 = a_2 = ... = a_n = 0$
$\therefore w = 0$
$\therefore W_j \cap \sum_{i \neq j} W_i = \{0\}$
$\therefore V = W_1 \oplus W_2 \oplus ... \oplus W_n$, W_i is the direct sum of 1-dimensional T-invariant subspaces.
$(\Leftarrow)$
$V = W_1 \oplus W_2 \oplus ... \oplus W_n$, 1-dimensional T-invariant subspace for $1 \leq i \leq n$.
Choose a nonzero vector $v_i \in W_i$ as the basis of W_i
$\forall w_i \in W$ & $w_i \neq 0$ let $w_i = a_i v_i, a_i \neq 0$
$T(w_i) \in W_i$ ($\because W_i$ is T-invariant)
We can let $T(w_i) = bv_i$
$\therefore T(w_i) = bv_i = (\frac{b}{a}) a v_i = \lambda_i w_i$, where $\lambda_i = \frac{b}{a}$
$\therefore w_i$ is an eigenvector of T.
Let $\sum_{i=1} a_i w_i = 0$

$$-a_j w_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} a_i w_i$$

But $-a_j w_j \in W_j$

$$\sum_{i \neq j} a_i w_i \in \sum_{i \neq j} W_i \ \& \ W_j \cap \sum_{i \neq j} W_i = \{0\}$$

$$\therefore -a_j w_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} a_i w_i = 0$$

$\therefore a_1 = a_2 = ... = a_n = 0$
$\therefore \{w_1, w_2, ..., w_n\}$ is independent and dim(V)=n.
$\Rightarrow \beta = \{w_1, w_2, ..., w_n\}$ can be a basis of V.
$\therefore \beta$ is a basis of V consisting of eigenvectors.
$\therefore$T is diagonalizable.

Sec. 6.1
6.1.5.
In C^2, show that $\langle x, y \rangle = xAy^*$ is an inner-product, where $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$ compute <x,

y> for x=(1-i, 2+3i) & y=(2+i, 3-2i)

Ans.:

(a) $\langle x, y\rangle = (1-i, 2+3i)\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}\begin{pmatrix} 2-i \\ 3+2i \end{pmatrix} = 6+21i$

(b)

(i) $\langle x+z, y\rangle = (x+z)Ay^* = xAy^* + zAy^* = \langle z, y\rangle + \langle z, y\rangle$

(ii) $\langle cx, y\rangle = (cx)Ay^* = c(xAy^*) = c\langle x, y\rangle$

(iii) Let $x = [x_1, x_2], y = [y_1, y_2]$

$$\overline{\langle x, y\rangle} = [x_1, x_2]\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}\begin{bmatrix} \overline{y_1} \\ \overline{y_2} \end{bmatrix} = [\overline{x_1}, \overline{x_2}]\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (\overline{x_1}y_1 + 2\overline{x_2}y_2) + i(\overline{x_2}y_1 - \overline{x_1}y_2)$$

$$\langle y, x\rangle = [y_1, y_2]\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \end{bmatrix} = (\overline{x_1}y_1 + 2\overline{x_2}y_2) + i(\overline{x_2}y_1 - \overline{x_1}y_2)$$

$$\langle y, x\rangle = [y_1, y_2]\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \end{bmatrix} = (\overline{x_1}y_1 + 2\overline{x_2}y_2) + i(\overline{x_2}y_1 - \overline{x_1}y_2)$$

$\therefore \overline{\langle x, y\rangle} = \langle y, x\rangle$

(iv) $\langle x, x\rangle = [x_1, x_2]\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \end{bmatrix} = (\overline{x_1}x_1 + 2\overline{x_2}x_2) + i(\overline{x_2}x_1 - \overline{x_1}x_2)$

$\because |x_{1-}x_2 i|^2 = x_1\overline{x_1} - \overline{x_1}x_2 i + \overline{x_2}x_2 + x_1\overline{x_2}i \geq 0$ and $\overline{x_2}x_2 \geq 0$

$\therefore \langle x, x\rangle \geq 0$

Q.E.D.

6.1.10.

Let V be an inner product space & suppose that x & y are orthogonal vectors in V.

Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagonean theorem in R^2.

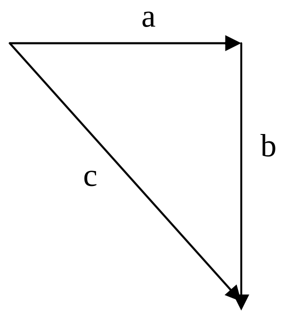
Ans.:

$\because x$ and y are orthogonal vector in V.

$\therefore \langle x, y\rangle = \langle y, x\rangle = 0$

$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \langle x,x\rangle + \langle y,y\rangle = \|x\|^2 + \|y\|^2$



Let *x* and *y* are vectors in R^2.

$\therefore$ *x* and *y* are orthogonal.

$\|x+y\|^2 = \|x\|^2 + \|y\|^2$

$\therefore c^2 = a^2 + b^2$

6.1.16.

(a) Show that the vector space H with $\langle \cdot, \cdot \rangle$ defined on page 32 is an inner product space.

(b) Let V=c([0,1]) & define $\langle f, g\rangle = \int_0^{1/2} f(t)g(t)dt$. Is this an inner product on V?

Ans.:

(a) $\because$ H is a complex function between [0, 2 π].

The inner product define as

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt$$

(i) $\langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle$

(ii) $\langle cf, g\rangle = c\langle f, g\rangle$

(iii) $\overline{\langle f, g\rangle} = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(t)\overline{g(t)}}dt = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(t)}g(t)dt = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt = \langle g, f\rangle$

(iv) $\langle f, f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{f(t)}dt = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 dt \geq 0$

$\therefore$ H is a inner product space.

(b)

(i) By definition, $\langle x, y\rangle$ is inner product, then $\langle x, x\rangle > 0$ if $x \neq 0$.(6.1)

(ii) for $\langle f, g\rangle = \int_0^{\frac{1}{2}} f(t)g(t)dt$

If $f(t) = \begin{cases} 0 & t \leq 0.5 \\ 1 & t > 0.5 \end{cases}$, then

$$\left\langle f, f \right\rangle = \int_0^{0.5} f(t) f(t) = 0, f(t) \neq 0$$

$\therefore \left\langle f, g \right\rangle = \int_0^{\frac{1}{2}} f(t) g(t) dt$ is not inner product.