Solution 8

Sec. 6.2

6.2.2.

In each part, apply the Gram-Schmidt process to the given subset S of the inner product space V to obtain an orthogonal basis for space(S). Then normalize the vectors in this basis to obtain an orthonormal basis β for space(S), and compute the Fourier coefficients of the given vector relative to β. Finally, use Theorem 6.5 to verify your result.

(h) V=$M_{2x2}(R)$, $S=\left\{\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix}\right\}$, and $A=\begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}$

Ans.:

(h) Let $S = \{v_1, v_2, v_3\}$ and orthogonal basis $\eta = \{w_1, w_2, w_3\}$, orthonormal basis $\beta = \{u_1, u_2, u_3\}$

$$v_1 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, v_2 = \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix}, v_3 = \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix} \text{and } \|v_1\| = \sqrt{13}$$

$$w_1 = v_1 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix},$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 = \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \right\rangle}{13} \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix} \text{and } \|w_2\| = 7$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2$$

$$= \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \right\rangle}{13} \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 4 & -12 \\ 3 & -16 \end{pmatrix}, \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix} \right\rangle}{49} \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & -8 \\ 7 & -14 \end{pmatrix} \text{ and } \|w_3\| = \sqrt{373}$$

$$\eta = \{u_1, u_2, u_3\} = \left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|} \right\} = \left\{ \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{1}{\sqrt{13}} \end{pmatrix}, \begin{pmatrix} \frac{5}{7} & \frac{-2}{7} \\ \frac{-4}{7} & \frac{2}{7} \end{pmatrix}, \begin{pmatrix} \frac{8}{\sqrt{373}} & \frac{-8}{\sqrt{373}} \\ \frac{7}{\sqrt{373}} & \frac{-14}{\sqrt{373}} \end{pmatrix} \right\}$$

Then verify it by Theorem 6.5

$$a_1 = \langle A, u_1 \rangle = \left\langle \begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{1}{\sqrt{13}} \end{pmatrix} \right\rangle = 5\sqrt{13}$$

$$a_1 = \langle A, u_2 \rangle = \left\langle \begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}, \begin{pmatrix} \frac{5}{7} & \frac{-2}{7} \\ \frac{-4}{7} & \frac{2}{7} \end{pmatrix} \right\rangle = -14$$

$$a_1 = \langle A, u_1 \rangle = \left\langle \begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}, \begin{pmatrix} \frac{8}{\sqrt{373}} & \frac{-8}{\sqrt{373}} \\ \frac{7}{\sqrt{373}} & \frac{-14}{\sqrt{373}} \end{pmatrix} \right\rangle = \sqrt{373}$$

$$A = \sum_{i=1}^{3} \langle A, u_i \rangle u_i$$

$$= 5\sqrt{13} \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{1}{\sqrt{13}} \end{pmatrix} - 14 \begin{pmatrix} \frac{5}{7} & \frac{-2}{7} \\ \frac{-4}{7} & \frac{2}{7} \end{pmatrix} + \sqrt{373} \begin{pmatrix} \frac{8}{\sqrt{373}} & \frac{-8}{\sqrt{373}} \\ \frac{7}{\sqrt{373}} & \frac{-14}{\sqrt{373}} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}$$

6.2.5.

Let $S_0=\{x_0\}$, where x_0 is a nonzero vector in R^3. Describe $S_0^{\perp}$ geometrically. Now suppose that $S=\{x_1, x_2\}$ is a linearly independent subset on R^3. Describe $S^{\perp}$ geometrically.

Ans.:

(i) $S^{\perp}$ is a set constructed by all vectors which are orthogonal with x_0. It's a plane which pass through the origin of the coordinates.

(ii) $S_0^{\perp}$ is a straight line which is orthogonal with a plane which is constructed by $\{x_1, x_2\}$.

6.2.14.

Let W_1 and W_2 be subspaces of finite-dimensional inner product space. Prove that $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$ and $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$. (See the definition of the sum of subsets of a vector space on page 22.) *Hint for the second equation: Apply Exercise 13(c) to the first equation.*

Ans.:

(a)

(i) $\forall x \in (\mathrm{W}_1 + \mathrm{W}_2)^{\perp}$

For any $w_1 \in \mathrm{W}_1$, $w_2 \in \mathrm{W}_2$

$x \perp (w_1 + w_2)$

$\therefore x \perp w_1 (w_2 = 0)$ and $x \perp w_2 (w_1 = 0)$

$\therefore x \perp \mathrm{W}_1$ and $\therefore x \perp \mathrm{W}_2$

$\therefore \mathrm{x} \in \mathrm{W}_1^{\perp} \cap \mathrm{W}_2^{\perp}$

$\therefore (\mathrm{W}_1 + \mathrm{W}_2)^{\perp} \subseteq \mathrm{W}_1^{\perp} \cap \mathrm{W}_2^{\perp}$

(ii) $\forall x \in \mathrm{W}_1^{\perp} \cap \mathrm{W}_2^{\perp}$

For any $y \in \mathrm{W}_1 + \mathrm{W}_2$, $y = y_1 + y_2$, where $y_1 \in \mathrm{W}_1$ and $y_2 \in \mathrm{W}_2$

$\therefore \langle x, y \rangle = \langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle = 0 \Rightarrow x \perp y$

$\therefore x \in (\mathrm{W}_1 + \mathrm{W}_2)^{\perp}$

$\therefore \mathrm{W}_1^{\perp} \cap \mathrm{W}_2^{\perp} \subseteq (\mathrm{W}_1 + \mathrm{W}_2)^{\perp}$

According to (i) and (ii), $(\mathrm{W}_1 + \mathrm{W}_2)^{\perp} = \mathrm{W}_1^{\perp} \cap \mathrm{W}_2^{\perp}$.

Q.E.D.

(b)

$\because (\mathrm{W}^{\perp})^{\perp} = \mathrm{W}$

$\therefore \mathrm{W}_1^{\perp} + \mathrm{W}_2^{\perp} = \left(\left(\mathrm{W}_1^{\perp} + \mathrm{W}_2^{\perp}\right)^{\perp}\right)^{\perp} = \left(\left(\mathrm{W}_1^{\perp}\right)^{\perp} \cap \left(\mathrm{W}_2^{\perp}\right)^{\perp}\right)^{\perp}$ (by (a))

$= \left(\mathrm{W}_1 \cap \mathrm{W}_1\right)^{\perp}$

$\therefore (\mathrm{W}_1 \cap \mathrm{W}_2)^{\perp} = \mathrm{W}_1^{\perp} + \mathrm{W}_2^{\perp}$

Q.E.D.

Sec. 6.3

6.3.2.

For each of the following inner product spaces V (over *F*) and linear transformations

g: $\mathrm{V} \to \mathrm{F}$, find a vector y such that g(*x*)= $\langle x, y \rangle$ for all $x \in \mathrm{V}$.

(c) V=P_2(R) with $\langle f, h \rangle = \int_0^1 f(t)h(t)dt, g(f) = f(0) + f'(1)$

Ans.:

We can get a orthonormal basis from Sec.6.2 exercise 2 is

$$\mathrm{P} = \{1, 2\sqrt{3}(x - \frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6})\}$$

$g(x_1) = g(1) = 1 + 0 = 1$

$$g(x_2) = g(2\sqrt{3}(x - \frac{1}{2})) = -\sqrt{3} + 2\sqrt{3} = \sqrt{3}$$

$$g(x_3) = g(6\sqrt{5}(x^2 - x + \frac{1}{6})) = \sqrt{5} + 6\sqrt{5} = 7\sqrt{5}$$

$$y = \sum_{i=1}^{3} g(x_i)x_i = 1 = 6(x - \frac{1}{2}) + 210(x^2 - x + \frac{1}{6}) = 210x^2 - 204x + 33$$

6.3.22.

Find the minimal solution to each of the following systems of linear equations.

(c) $\begin{array}{l} x + y - z = 0 \\ 2x - y + z = 3 \\ x - y + z = 2 \end{array}$

Ans.:

(c)

$$\text{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$\text{AA}^* = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 6 & 4 \\ -1 & 4 & 3 \end{bmatrix}$$

$$\begin{array}{r} 3x - z = 0 \\ 6y + 4z = 3 \\ -x + 4y + 3z = 2 \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0 \end{bmatrix}$$

$$\text{A}^* \begin{bmatrix} 0 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix}$$

$$\Rightarrow x = 1, y = -0.5, z = 0.5$$

Sec. 6.4

6.4.2

For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of V and list the corresponding eigenvalues.

(b) V=R^3 and T is defined by T(a, b, c)=(-a+b, 5b, 4a-2b+5c).

Ans.:

(b)

(i) $\beta = \{e_1, e_2, e_3\}$, $\mathrm{T}(e_2) = (1,5,-2)$

$$[\mathrm{T}]_\beta = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & 0 \\ 4 & -2 & 5 \end{bmatrix}, \quad [\mathrm{T}]_\beta^* = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 5 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$\because [\mathrm{T}]_\beta \neq [\mathrm{T}]_\beta^*$ $\therefore$ Not self-adjoint.

$$[\mathrm{T}]_\beta [\mathrm{T}]_\beta^* = \begin{bmatrix} 2 & 5 & -6 \\ 5 & 25 & -10 \\ -6 & -10 & 45 \end{bmatrix}$$

$$[\mathrm{T}]_\beta^* [\mathrm{T}]_\beta = \begin{bmatrix} 17 & -9 & 20 \\ -9 & 30 & -10 \\ 20 & -10 & 25 \end{bmatrix}$$

$\because [\mathrm{T}]_\beta [\mathrm{T}]_\beta^* \neq [\mathrm{T}]_\beta^* [\mathrm{T}]_\beta$ $\therefore$ Not normal.

$\therefore$ T is neither.

(ii) $\det\left(\begin{bmatrix} -1-\lambda & 1 & 0 \\ 0 & 5-\lambda & 0 \\ 4 & -2 & 5-\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda = -1,5,5$

Eigenvalues are -1, 5, 5.

(iii) $\lambda = 5 \Rightarrow \text{eigenvector} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = -1 \Rightarrow \text{eigenvector} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$

Then orthonormal basis of eigenvectors of T is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3/\sqrt{13} \\ 0 \\ -2/\sqrt{13} \end{bmatrix} \right\}$

6.4.4.

Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if TU=UT.

Ans.:

$(\Rightarrow)$

Suppose that TU is self-adjoint.

$\because (TU)^* = U^*T^* = UT$

$\therefore (TU)^* = TU$

$\Rightarrow UT = TU$

$(\Leftarrow)$

Suppose that TU=UT then $(TU)^* = U^*T^* = UT = TU$

$\therefore TU$ is self-adjoint.