Repeat–Accumulate & Irregular Repeat–Accumulate Codes

Lecture 4：Part B

Contents

Introduction (Cont.)

- The final problem for channel coding researchers: For a given channel, find an ensemble of codes with：
  - (1) a linear-time encoding algorithm
  - (2) which can be decoded reliably in linear time at rates arbitrarily close to channel capacity.
- For turbo-codes, on the AWGN channel there appears to be a gap between channel capacity and the iterative decoding thresholds for any turbo ensemble.

Introduction (Cont.)

- For LDPC codes,
  - the natural encoding algorithm is quadratic in the block length.
  - for regular LDPC codes, on the binary symmetric and AWGN channels there is a gap between capacity and the iterative decoding threshold.
  - on the binary erasure channel irregular LDPC codes satisfy (2).
  - on the AWGN channel, irregular LDPC codes are markedly better than regular ones, but whether or not they can reach capacity is not yet known.

Introduction

- For repeat-accumulated (RA) codes (including regular or irregular one),
  - the natural encoding algorithm is linear in the block length.
  - for regular RA codes, on the binary symmetric and AWGN channels there is a gap between capacity and the iterative decoding threshold.
  - on the binary erasure channel irregular RA (IRA) codes satisfy (2).
  - on the AWGN channel, IRA codes are markedly better than regular ones, but whether or not they can reach capacity is not yet known.

RA Code Structure

$[x_1, x_2, x_3, \ldots.]$

$[x_1, x_1+x_2, x_1+x_2+x_3, \ldots.]$

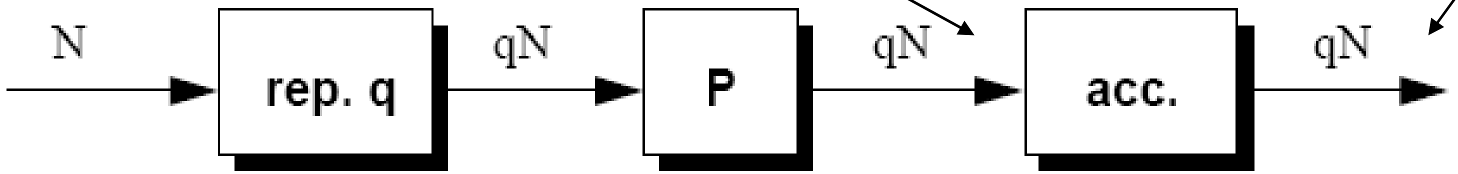
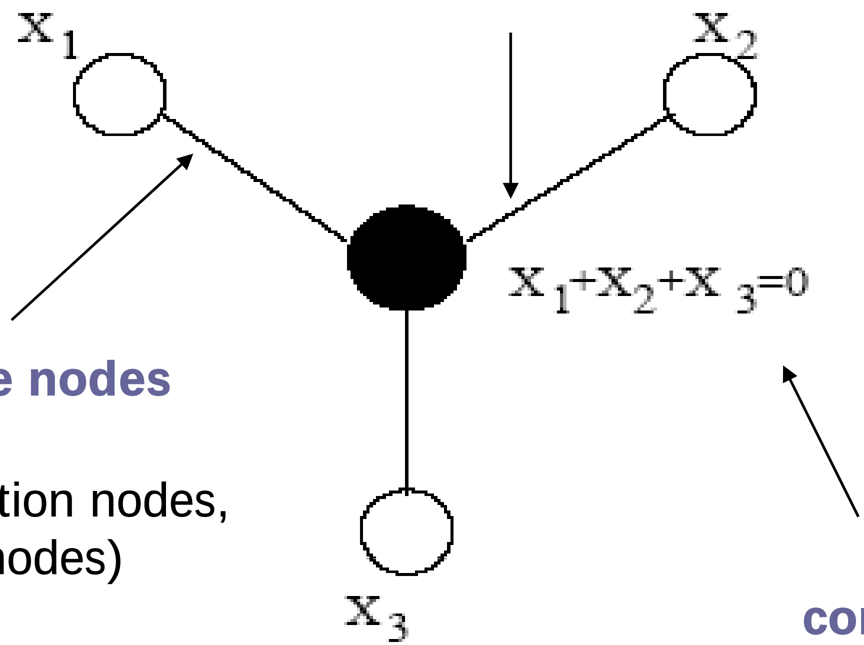


Figure 3.1: Encoder for a (qN, N) RA code. The "rep. q" component repeats its N-bit input block q times; the "P" block represents an arbitrary permutation of its qN-bit input block; and the "acc." is an accumulator, whose outputs are the mod-2 partial sums of its inputs.

interleaving

Tanner graph representation

Check nodes

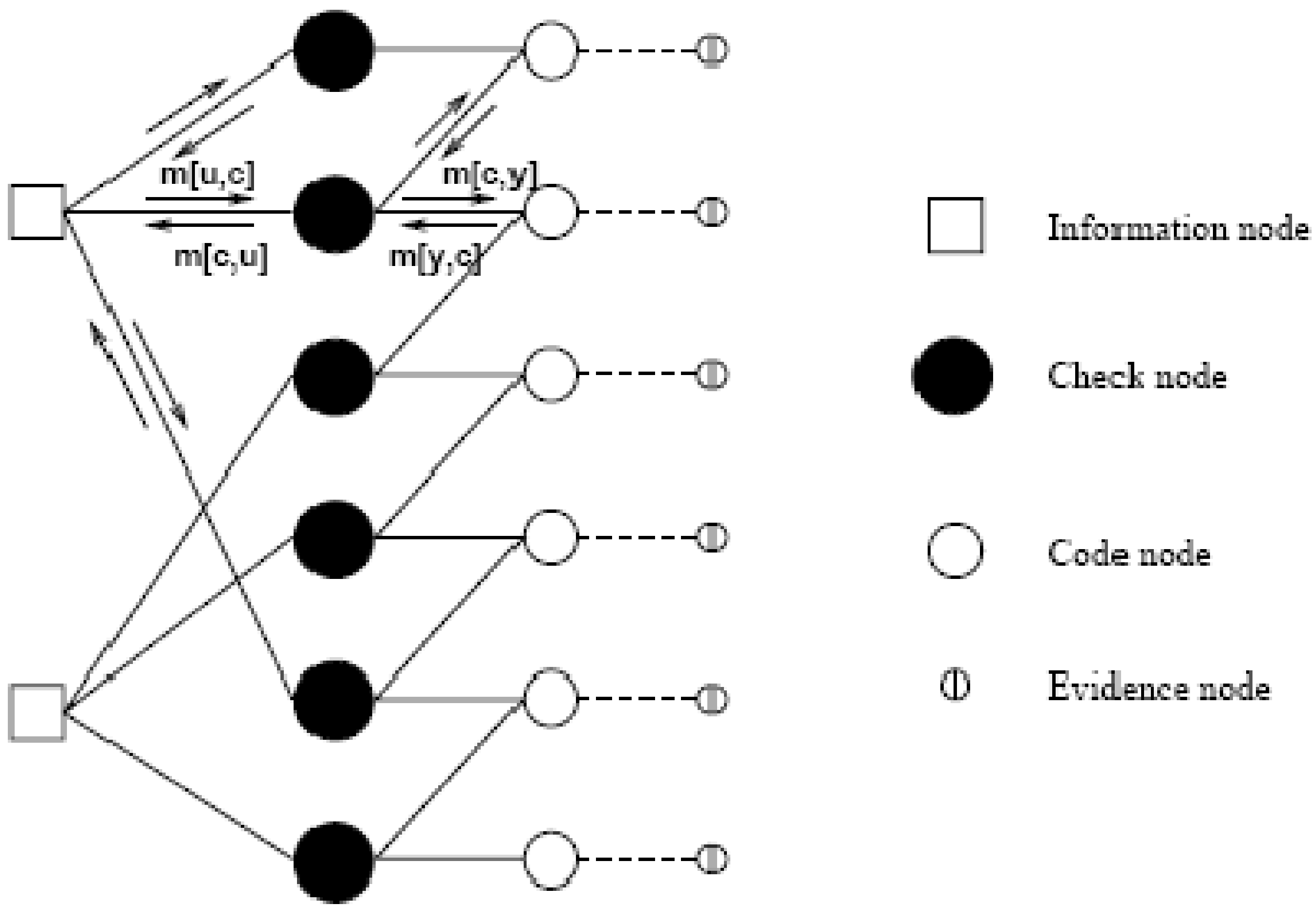


Variable nodes

(information nodes,
parity nodes)

constraints

Iterative Decoding of RA codes



Iterative Decoding of RA codes (cont.)

- *m[u, c]*: information from *u* (information node) to *c* (check node)
- *m[c, u]* :information from *c* to *u*
- *m[y, c]* :information from *y* (code node) to *c*
- *m[c, y]* :information from *c* to *y*

Iterative Decoding of RA codes (cont.)

- $\textbf{\textit{Update}}\ \mathbf{m[y, c]}$:

$$m[y, c] = \begin{cases} B(y) & \text{if } y = y_{qn}, \\ B(y) + m[c', y] & \text{otherwise, where } (c', y) \in E \text{ and } c' \neq c. \end{cases}$$

- $\textbf{\textit{Update}}\ \mathbf{m[u, c]}$:

$$m[u, c] = \sum_{c'} m[\text{C',u}], \ \text{where } (u, c') \in E \text{ and } c' \neq c.$$

Iterative Decoding of RA codes (cont.)

$$L(u) = \log \frac{P(u=0)}{P(u=1)}$$

$$P(u_1 \oplus u_2 = 0) = P(u_1 = 0)P(u_2 = 0) + P(u_1 = 1)P(u_2 = 1)$$

$$P(u_1 \oplus u_2 = 1) = P(u_1 = 1)P(u_2 = 0) + P(u_1 = 0)P(u_2 = 1)$$

$$P(u=0) = \frac{P(u=1)}{P(u=1)+P(u=0)} = \frac{P(u=0)\big/P(u=1)}{1+P(u=0)\big/P(u=1)} = \frac{e^{L(u)}}{1+e^{L(u)}}$$

$$\therefore L(u_1 \oplus u_2) = \log \frac{P(u_1 \oplus u_2 = 0)}{P(u_1 \oplus u_2 = 1)}$$

$$= \log \frac{e^{L(u_1)}\big/\left(1+e^{L(u_1)}\right)\, e^{L(u_2)}\big/\left(1+e^{L(u_2)}\right) + 1\big/\left(1+e^{L(u_1)}\right)\, 1\big/\left(1+e^{L(u_2)}\right)}{e^{L(u_1)}\big/\left(1+e^{L(u_1)}\right)\, 1\big/\left(1+e^{L(u_2)}\right) + 1\big/\left(1+e^{L(u_1)}\right)\, e^{L(u_2)}\big/\left(1+e^{L(u_2)}\right)} = \log \frac{1+e^{L(u_1)+L(u_2)}}{e^{L(u_1)}+e^{L(u_2)}}$$

Iterative Decoding of RA codes (cont.)

- ***Update at check nodes,* m[c, y] : *and* m[c, u] :**

$$m[c,y] = \begin{cases} m[u,c] & \text{if } c = c_1, \text{ where } (u,c) \in E \text{ and } u \in U, \\ \log \dfrac{1+e^{m[u,c]+m[y',c]}}{e^{m[u,c]}+e^{m[y',c]}} & \text{otherwise, where } (u,c),(y',c) \in E \text{ and } y \neq y' \in Y. \end{cases}$$

$$m[c,u] = \begin{cases} m[y,c] & \text{if } c = c_1, \text{ where } (y,c) \in E \text{ and } y \in Y, \\ \log \dfrac{1+e^{m[y,c]+m[y',c]}}{e^{m[u,c]}+e^{m[y',c]}} & \text{otherwise, where } (y,c),(y',c) \in E \text{ and } y \neq y' \in Y. \end{cases}$$

Performance of RA codes on AWGN

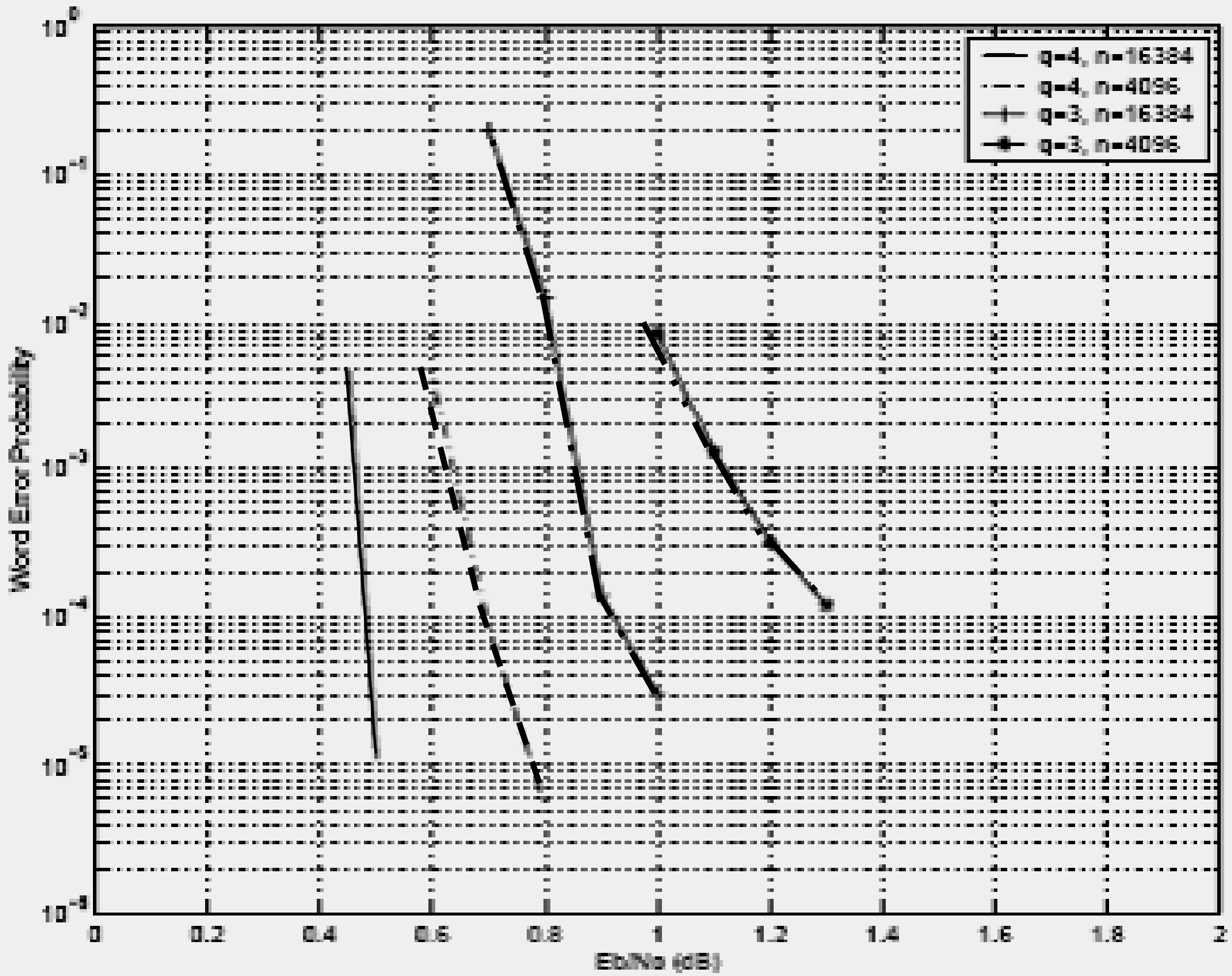


Figure 3.5: Simulated performance of iterative decoding of RA codes on an AWGN channel.

Definition of IRA Codes

- $f_i \geqq 0$, $\Sigma f_i = 1$, a is a positive integer.
- Two kinds of nodes
  - k information nodes and r parity nodes
  - r check nodes

$$ra = k \times \sum_i (i \times f_i)$$

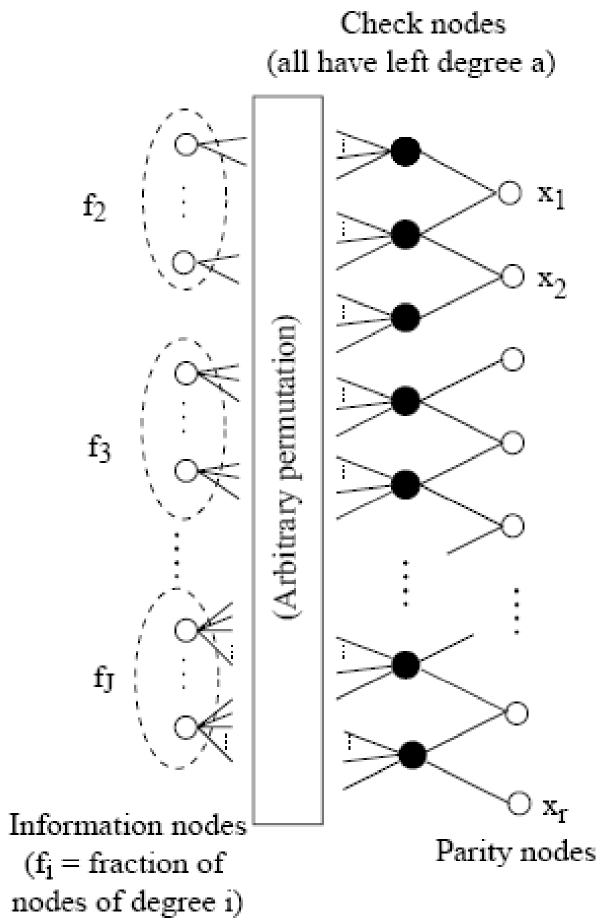


Figure 1: Tanner graph for IRA code with parameters $(f_1, \ldots, f_J; a)$.

Definition of IRA Codes (cont.)

- Encoding complexity is O(n).

$$x_j = x_{j-1} + \sum_{i=1}^{a} v_{(j-1)a+i},$$

- Two versions of IRA code
  - Nonsystematic
    - $(u_1, \ldots, u_k) \Rightarrow (x_1, \ldots, x_r)$
  - Systematic
    - $(u_1, \ldots, u_k) \Rightarrow (u_1, \ldots, u_k; x_1, \ldots, x_r)$

Definition of IRA Codes (cont.)

- For nonsystematic codes (if a=1,f_i=1&f_k=0 for $k \neq i$, it is RA codes)

$$R_{nsys} = \frac{a}{\sum_i i \times f_i} = \frac{a}{ra/k} = \frac{k}{r}$$

- For systematic codes

$$R_{sys} = \frac{a}{a + \sum_i i \times f_i} = \frac{a}{a + ra/k} = \frac{ka}{a(k+r)} = \frac{k}{k+r}$$

Structure of IRA codes

A closer look of the structure of IRA codes reveals that it is a serial concatenation of two codes, low-density generator matrix (LDGM) code and accumulate code

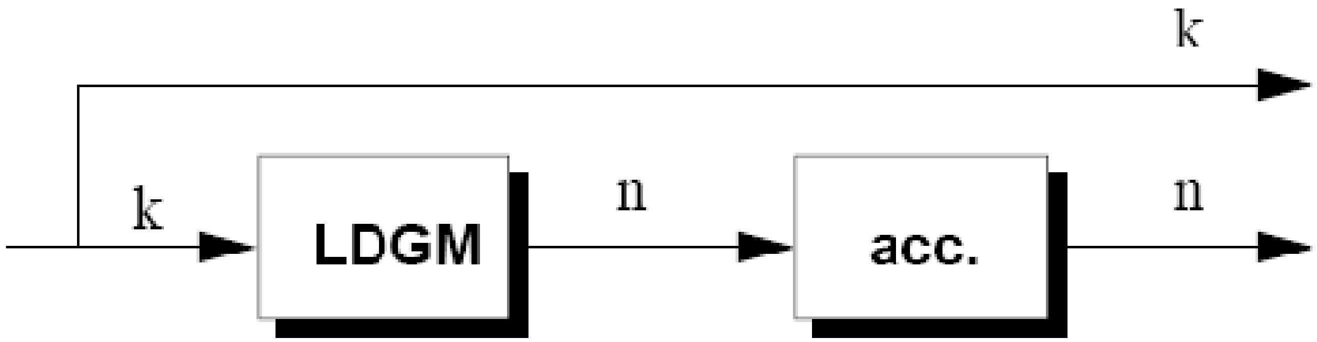


Figure 7.2: IRA code as a serial turbo code.

Iterative Decoding of IRA Codes

message from check nodes to variable nodes

$$m(u \to v) = \sum_{w \neq v} m(w \to u) + m_0(u),$$

$m_0(u)$ is LLR message associated with channel observation of the codeword u

message from variable nodes to check nodes

$$\tanh \frac{m(u \to v)}{2} = \prod_{w \neq v} \tanh \frac{m(w \to u)}{2}.$$

Performance of IRA codes

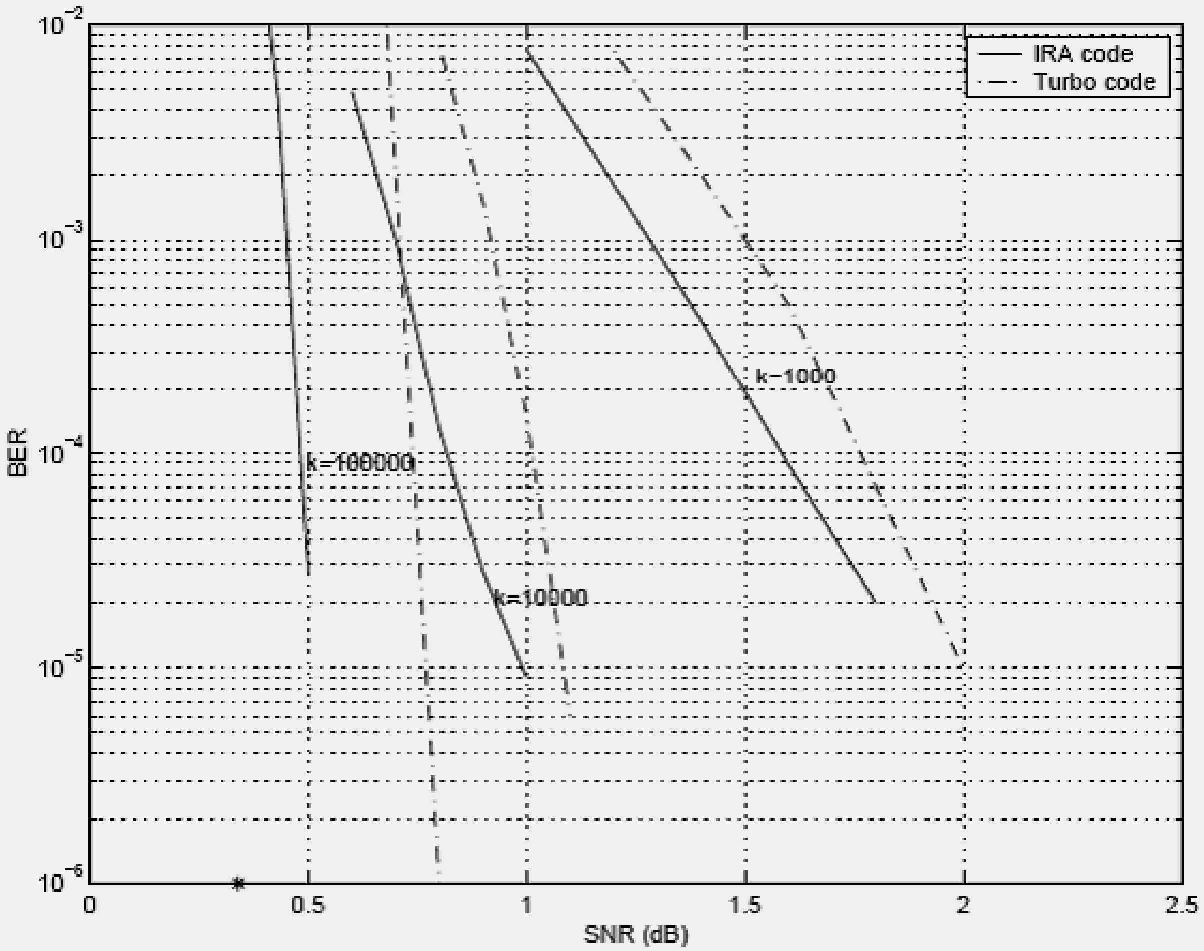
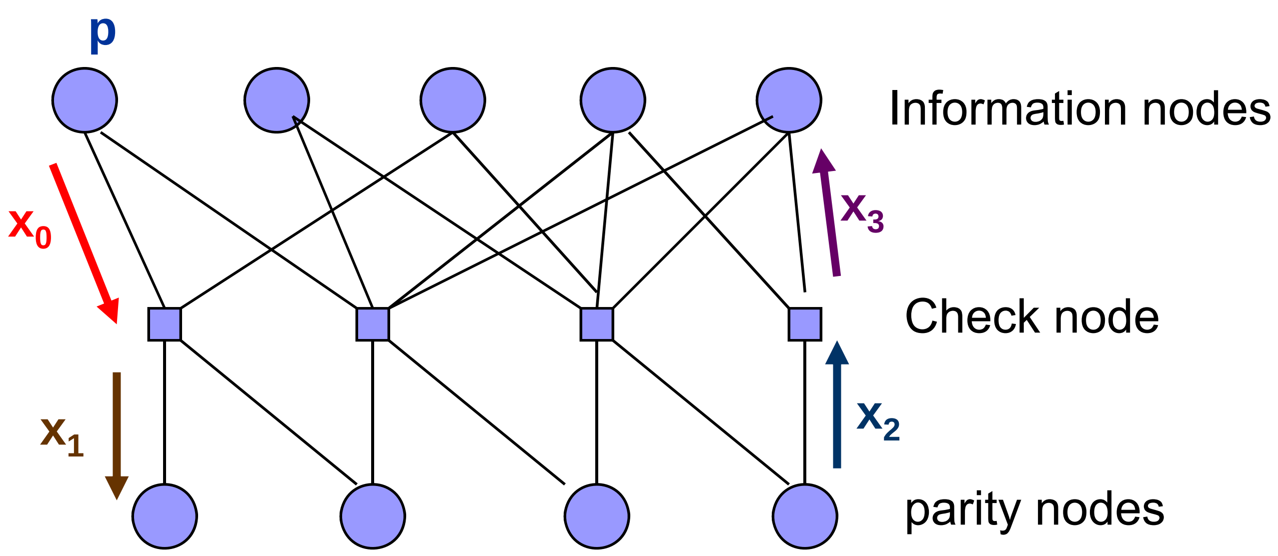


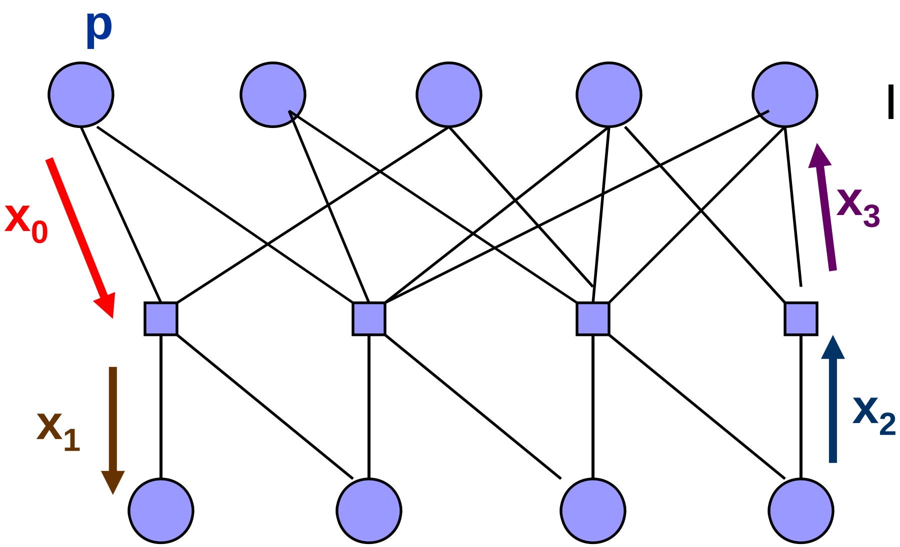
Figure 7.3: Comparison between turbo codes (dashed curves) and IRA codes (solid curves) of lengths $k = 10^3, 10^4, 10^5$. All codes are of rate one-half.

Iterative Decoding analysis of IRA codes



- x0：The probability of erasure on an edge from an information node to a check node
- x1：The probability of erasure on an edge from a check node to a parity node
- x2：The probability of erasure on an edge from a parity node to a check node
- x3：The probability of erasure on an edge from a check node to an information node
- p：The initial probability of erasure on the message bits

Iterative Decoding analysis of IRA codes (cont.)



$$R(x) = \frac{\int_0^x \rho(t)dt}{\int_0^1 \rho(t)dt}.$$

$$x_1 = 1 - (1 - x_2) R(1 - x_0),$$

$$x_2 = p x_1,$$

$$x_3 = 1 - (1 - x_2)^2 \rho(1 - x_0), \Rightarrow \quad p\lambda\left(1 - \left[\frac{1-p}{1-pR(1-x)}\right]^2 \rho(1-x)\right) = x.$$

$$x_0 = p\lambda(x_3).$$

Iterative Decoding analysis of IRA codes (cont.)

$$p\lambda\left(1-\left[\frac{1-p}{1-pR(1-x)}\right]^2\rho(1-x)\right)=x.$$

If this equation has no solution in the interval (0, 1], then iterative decoding must converge to probability of erasure zero.

Grouping factor : a $\Rightarrow$ $\rho(x)=x^{a-1}, R(x)=x^a$

$$p\lambda\left(1-\left[\frac{1-p}{1-pR(1-x)}\right]^2\rho(1-x)\right)<x, \quad \forall x\neq 0,$$

Iterative Decoding of IRA codes

From now on we will use a special algebra for the log-likelihood ratio values $L(u)$: We use the symbol $\boxplus$ as the notation for the addition defined by

$$L(u_1) \boxplus L(u_2) \triangleq L(u_1 \oplus u_2)$$

with the additional rules

$$L(u) \boxplus \infty = L(u) \qquad L(u) \boxplus -\infty = -L(u)$$

and

$$L(u) \boxplus 0 = 0.$$

Box-plus operation(田)

Information: log-likelihood ratio

C＝A田B⇔C＝ $\log[\frac{1+e^{A+B}}{e^{A}+e^{B}}]$

田 : box-plus operation

Iterative Decoding of IRA codes

$$\sum_{j=1}^{J} \boxplus L(u_j) = L(\sum_{j=1}^{J} \boxplus u_j) = \log \frac{\prod_{j=1}^{J}(e^{L(uj)}+1)+\prod_{j=1}^{J}(e^{L(uj)}-1)}{\prod_{j=1}^{J}(e^{L(uj)}+1)-\prod_{j=1}^{J}(e^{L(uj)}-1)}$$

$$= \log \frac{1+\prod_{j=1}^{J}(\tanh\frac{u_j}{2})}{1-\prod_{j=1}^{J}(\tanh\frac{u_j}{2})} \xrightarrow{\text{令}\log\frac{1+\prod\cdots}{1-\prod\cdot\cdot}=u} \tanh\frac{u}{2} = \frac{e^u-1}{e^u+1}$$

$$\Rightarrow \tanh\frac{1}{2}\log\frac{1+x}{1-x} = \frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1} = \frac{2x}{2} = x$$

$$\Rightarrow \tanh\frac{1}{2}\log\frac{1+\prod_{j=1}^{J}(\tanh\frac{u_j}{2})}{1-\prod_{j=1}^{J}(\tanh\frac{u_j}{2})} = \prod_{j=1}^{J}(\tanh\frac{u_j}{2})$$