Solution 1

Sec. 1.2
4.
(a) \[
\begin{bmatrix}
2 & 5 & -3 \\
1 & 0 & 7 \\
\end{bmatrix}
+ \begin{bmatrix}
4 & -2 & 5 \\
-5 & 3 & 2 \\
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
2 & 5 & -3 \\
1 & 0 & 7 \\
\end{bmatrix}
= \begin{bmatrix}
-701 & 352 \\
236 & -934 \\
\end{bmatrix}
\]
Ans.:
(a) \[
\begin{bmatrix}
6 & 3 & 2 \\
-4 & 3 & 9 \\
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
8 & 20 & -12 \\
4 & 0 & 28 \\
\end{bmatrix}
\]

12.
A real-valued function \( f \) defined on the real line is called an even function if \( f(-t) = f(t) \) for each real number \( t \). Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.
Ans.:
Let \( V \) denote the set of even functions.
(1) Closure of vector addition:
For all \( f(t), g(t) \) in \( V \), define \( l(t) = f(t) + g(t) \)
\[ l(t) = f(t) + g(t) = f(-t) + g(-t) = l(-t) \]
\( \therefore l(t) \in V \)
(2) Closure of scalar multiplication:
For each element \( a \) in \( F \) and all \( f(t) \) in \( V \),
\( a(f(t)) = a(f(-t)) \in V \)
(3)
For all \( f(t), g(t) \) in \( V \), \( f(t) + g(t) = g(t) + f(t) \) (commutativity of addition)
\( \Rightarrow (VS \ 1) \) holds.
(4)
For all \( f(t), g(t) \) and \( h(t) \) in \( V \), \( f(t) + (g(t) + h(t)) = (f(t) + g(t)) + h(t) \) (associativity of addition)
\( \Rightarrow (VS \ 2) \) holds.
(5)
Define \( w(t) \) by \( w(t) = 0 \) for all \( t \).
\( \therefore w(t) = w(-t) = 0 \)
18. Let \( V = \{ (a_1, a_2) : a_1, a_2 \in R \} \). For \((a_1, a_2), (b_1, b_2) \in V \) and \( c \in R \), define \((a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \) and \( c(a_1, a_2) = (ca_1, ca_2) \).
Is \( V \) a vector space over \( R \) with these operations? Justify your answer.

\[
(a_1 + b_1, a_2 + 2b_2) \neq (a_1 + 2a_1, b_2 + 3a_2) \quad \text{(VS 1) fails to hold.}
\]

\[
\therefore V \text{ is not a vector space.}
\]
Ans.:
Let the entry of A that lies in row $i$ and column $j$ be $a_{ij}$. Then the entry of $A^t$ that lies in row $i$ and column $j$ is $a_{ji}$.

⇒ The entry of matrix $(A + A^t)$ that lies in row $i$ and column $j$ is $a_{ij} + a_{ji}$. Similarly, $a_{ji} + a_{ij}$ lies in row $j$ and column $i$ of matrix $(A + A^t)$.

∴ $a_{ij} + a_{ji} = a_{ji} + a_{ij}$

⇒ The entry of matrix $(A + A^t)$ that lies in row $i$ and column $j$ is equal to the entry of matrix $(A + A^t)$ that lies in row $j$ and column $i$.

$(A + A^t)$ is symmetric.

10. Prove that $W_1 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_1 + a_2 + \ldots + a_n = 0\}$ is a subspace of $F^n$, but $W_2 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_1 + a_2 + \ldots + a_n = 1\}$ is not.

Ans.:

(I)
(i) For any two vector $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $W_1$,

\[ x_1 + x_2 + \ldots + x_n = 0 \]
\[ y_1 + y_2 + \ldots + y_n = 0 \]
\[ x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \]
\[ \Rightarrow (x_1 + y_1) + (x_2 + y_2) + \ldots + (x_n + y_n) = 0 \]
\[ \Rightarrow x + y \in W_1 \]

(ii) For any $c \in F$

\[ cx = c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n) \]
\[ \Rightarrow cx_1 + cx_2 + \ldots + cx_n = c(x_1 + x_2 + \ldots + x_n) = c \cdot 0 = 0 \]
\[ \Rightarrow cx \in W_1 \]

(iii) For zero vector $z = (0, 0, \ldots, 0) \in F^n$

\[ \Rightarrow 0 + 0 + \ldots + 0 = 0 \]
\[ \Rightarrow z \in W_1 \]

Base on (i), (ii) and (iii), $W_1$ is a subspace of $F^n$.

(II)
(i) For any two vector $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $W_2$, then

\[ x_1 + x_2 + \ldots + x_n = 1 \]
\[ y_1 + y_2 + \ldots + y_n = 1 \]
\[ x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \]
\[ \Rightarrow (x_1 + y_1) + (x_2 + y_2) + \ldots + (x_n + y_n) \]
\[
= (x_1 + x_2 + \ldots + x_n) + (y_1 + y_2 + \ldots + y_n)
= 1 + 1 = 2 \neq 1
\]
\[\therefore x + y \notin W_2\]

(ii) For any \(c \in F\) is a constant
\[
\begin{align*}
    cx &= c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n) \\
    \Rightarrow cx_1 + cx_2 + \ldots + cx_n &= c(x_1 + x_2 + \ldots + x_n) = c \cdot 1 = c
\end{align*}
\]
\[\therefore cx \notin W_2\]

(iii) For zero vector \(z = (0,0,\ldots,0) \in F^n\)
\[\therefore 0 + 0 + \ldots + 0 = 0 \neq 1\]
\[\therefore z \notin W_1\]

According to either (i) or (ii) or (iii), \(W_2\) is not a subspace of \(F^n\).

19.
Let \(W_1\) and \(W_2\) be subspaces of a vector space \(V\). Prove that \(W_1 \cup W_2\) is a subspace of \(V\) if and only if \(W_1 \subseteq W_2\) or \(W_2 \subseteq W_1\).

Ans.:
\((\Leftarrow)\) Suppose \(W_1 \subseteq W_2\) or \(W_2 \subseteq W_1\), then \(W_1 \cup W_2 = W_2\) or \(W_1\),
\[\therefore W_2\] and \(W_1\) are subspace of \(V\).
\[\therefore W_1 \cup W_2\] is a subspace of \(V\).

\((\Rightarrow)\) Assume \(W_1 \cup W_2\) is a subspace of \(V\):
Suppose that neither \(W_1 \subseteq W_2\) nor \(W_2 \subseteq W_1\) is true.
We can find two vectors \(x\) and \(y\) which satisfy \(x \in W_1, x \notin W_2\) and \(y \in W_2, y \notin W_1\)
\((x \neq y \neq 0)\).
\[\therefore x \in W_1\] and \(y \in W_2\).
\[\therefore x \text{ and } y \in W_1 \cup W_2\]
\[\therefore W_1 \cup W_2\] is a subspace of \(V\).
\[\therefore x + y \in W_1 \cup W_2\]
\[\Rightarrow x + y \in W_1 \text{ or } W_2\]

a) Suppose \(x + y \in W_1\):
\[\therefore x \in W_1\]
\[\therefore x \in W_1\]
\[\Rightarrow (x + y) + (-x) \in W_1\]
\[\therefore y \in W_1\Rightarrow conflicting\]

b) Suppose \(x + y \in W_2\)
\[\therefore y \in W_1\]
\[\therefore y \in W_2\]
⇒ (x + y) + (-y) ∈ W₂ ⇒ x ∈ W₂ ⇒ conflicting
⇒ The assumption of “neither W₁ ⊆ W₂ nor W₂ ⊆ W₁ is true” fails.
∴ W₁ ⊆ W₂ or W₂ ⊆ W₁.

Base on (i) and (ii), W₁ ∪ W₂ is a subspace of V if and only if W₁ ⊆ W₂ or W₂ ⊆ W₁.

Q.E.D.

Sec. 1.4
5.
In each part, determine whether the given vector is in the span of S.
(b) (-1, 2, 1), S = {(1,0,2), (-1,1,1)}
(g) \[
\begin{bmatrix}
1 & 2 \\
-3 & 4
\end{bmatrix}
\]
S = \[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 1
\end{bmatrix}
\]

Ans.:
(b) (-1, 2, 1) isn’t in the span of S because there are’t any solution for
a(1,0,2)+b(-1,1,1)=(-1, 2, 1)

(g) \[
\begin{bmatrix}
1 & 2 \\
-3 & 4
\end{bmatrix}
\]
is in the span of S because \[
\begin{bmatrix}
1 & 2 \\
-3 & 4
\end{bmatrix}
\] = \[
\begin{bmatrix}
1 & 0 \\
-1 & 0
\end{bmatrix}
\] + \[
\begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix}
\] - \[
\begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}
\]

13.
Show that if S₁ and S₂ are subsets of a vector space V such that S₁ ⊆ S₂ then span(S₁) ⊆ span(S₂). In particular, if S₁ ⊆ S₂ and span(S₁) = V, deduce that span(S₂) = V.

Ans.:
(A)
Suppose that S₁ = \{x₁, x₂, ..., xₘ\} is a subset of a vector space V
⇒ Span(S₁) and Span(S₂) are subspace of V
For all x ∈ Span(S₁), x = a₁x₁ + a₂x₂ + ... + aₘxₘ
∴ S₁ ⊆ S₂
∴ \{x₁, x₂, ..., xₘ\} ∈ Span(S₂)
⇒ x = a₁x₁ + a₂x₂ + ... + aₘxₘ ∈ S₂
⇒ Span(S₁) ⊆ Span(S₂)

(B)
According to (A), if S₁ ⊆ S₂ then Span(S₁) ⊆ Span(S₂).
∴ Span(S₁) = V
⇒ V ⊆ Span(S₂)
According to Theorem 1.5, if S₂ is a subset of the vector space V then Span(S₂) ⊆ V
∴ Span(S₂) = V